

# AVERAGE EQUIDISTRIBUTION OF HEEGNER POINTS ASSOCIATED TO THE 3-PART OF THE CLASS GROUP OF IMAGINARY QUADRATIC FIELDS

BOB HOUGH

**ABSTRACT.** We prove that the Heegner points attached to the 3-part of the class group of an imaginary quadratic field  $\mathbb{Q}(\sqrt{-d})$  equidistribute in  $\mathcal{F} = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$  on average over  $d$  as  $d \rightarrow \infty$ . As a consequence, we obtain a proof of the Davenport-Heilbronn theorem on the mean size of the 3-part of the class group without first passing through cubic fields. We also prove a uniform vertical density of Heegner points associated to the  $k$ -part of the class group high in the cusp of  $\mathcal{F}$ , for any odd  $k$ . This leads to a conjectural negative secondary main term in the mean size of the  $k$ -part of the class group, refining the prediction of the Cohen-Lenstra heuristic.

## 1. INTRODUCTION

Let  $d > 0$  be squarefree and let  $H_3(-d)^*$  be the non-principal 3-part of the class group of  $\mathbb{Q}(\sqrt{-d})$ , that is, classes of ideals  $\mathfrak{a}$  for which  $\mathfrak{a}^3$  is principal but  $\mathfrak{a}$  is not. In [6] Davenport and Heilbronn proved that the average size of  $H_3(-d)^*$  is 1 for  $0 < d < D$ , which agrees with the Cohen-Lenstra random model [3] for statistics of the class group. Here we extend the “average random nature” of the ideals of  $H_3(-d)^*$  by showing that they verify Duke’s equidistribution result for Heegner points [7].

Recall that an ideal  $\mathfrak{a}$  is said to be primitive if it has no integer divisors larger than 1. A primitive ideal  $\mathfrak{a}$  may be written as a  $\mathbb{Z}$ -module in the form  $\mathfrak{a} = [Na, b + \sqrt{-d}]$  if  $d \equiv 1, 2 \pmod{4}$ ,  $\mathfrak{a} = [Na, b + \frac{1+\sqrt{-d}}{2}]$  if  $d \equiv 3 \pmod{4}$ , with  $\frac{-Na}{2} < b \leq \frac{Na}{2}$ ; the Heegner point  $z_{\mathfrak{a}}$  attached to  $\mathfrak{a}$  is the ratio between the two generators. In [7] Duke shows that for a given  $d$  the Heegner points associated to primitive ideals of  $\mathbb{Q}(\sqrt{-d})$  become equidistributed in the upper half plane with respect to hyperbolic measure, as  $d \rightarrow \infty$ . The corresponding statement cannot hold for the Heegner points of  $H_3(-d)^*$  for a single growing  $d$  since the size of  $H_3(-d)$  is typically bounded, but, in analogy with the Davenport-Heilbronn Theorem (D-H), our principal result says that equidistribution holds on average.

**Theorem 1.1.** *Let  $B$  be a bounded Borel measurable subset of the upper half plane  $\mathbb{H}$  having boundary of measure zero. Then as  $D \rightarrow \infty$ ,*

$$\sum_{\substack{0 < -d < -D \\ d \equiv 2 \pmod{4} \\ \text{squarefree}}} \#\{\mathfrak{a} \text{ primitive} : [\mathfrak{a}] \in H_3(-d)^*, z_{\mathfrak{a}} \in B\} \sim \frac{6D}{\pi^3} \mathrm{vol}(B)$$

In this sense, the ideals of  $H_3(-d)^*$  are not distinguished from the general ideals of the class group.

Our proof of Theorem 1.1 builds on Soundararajan’s work in [14] on divisibility of the class number, in which he counts solutions to a diophantine equation parametrizing primitive ideals of fixed order. For ideals of order three we give an asymptotic for the number of ideals having Heegner point lying above a horizontal line and then isolate the real part of the Heegner point. This yields Theorem 1.1 as a consequence of the following separate vertical and horizontal equidistribution calculations.

**Theorem 1.2.** For  $D^{-1/6} < Y$ ,

$$\mathcal{S}(D, Y) := \sum_{\substack{0 > -d > -D \\ d \equiv 2 \pmod{4} \\ \text{squarefree}}} \sum_{\substack{\alpha \text{ primitive} \\ [\alpha] \in H_3(-d)^* \\ \mathfrak{I}(z_\alpha) > \frac{1}{Y}}} 1 = \frac{6}{\pi^3} Y D + C_{5/6} D^{5/6} \\ + O(Y^{7/4} D^{7/8+\epsilon} + D^{5/6} \exp(-c(\log Y^{3/2} D^{1/4})^{1/3}) + Y^3).$$

Here

$$C_{5/6} = \frac{2\zeta(\frac{1}{3})\Gamma(\frac{1}{6})}{5\pi^{\frac{3}{2}}\Gamma(\frac{2}{3})} \left[1 - 2^{\frac{1}{3}} + 2^{\frac{2}{3}}\right] \prod_{p \text{ odd}} \left(1 - \frac{p^{\frac{1}{3}} + 1}{p^2 + p}\right).$$

**Theorem 1.3.** For  $D^{-1/6} < Y < D^{1/6}$  and  $f \in \mathbb{Z} \setminus \{0\}$ ,

$$\mathcal{S}(D, Y; f) := \sum_{\substack{0 > -d > -D \\ d \equiv 2 \pmod{4} \\ \text{squarefree}}} \sum_{\substack{\alpha \text{ primitive} \\ [\alpha] \in H_3(-d)^* \\ \mathfrak{I}(z_\alpha) > \frac{1}{Y}}} e(f\Re(z_\alpha)) \\ = O(f^{1/4} Y^{11/8} D^{15/16+\epsilon} + f^{1/2} Y^{5/4} D^{7/8+\epsilon} + f Y^{3/4} D^{7/8+\epsilon}).$$

For fixed  $f$  and  $D^{-1/6+\epsilon} < Y < D^{1/6-\epsilon}$ ,  $\mathcal{S}(D, y; f) = o(\mathcal{S}(D, y))$ .

Theorem 1.2 says that asymptotically the right number of Heegner points lie above the horizontal line  $\mathfrak{I}(z) > \frac{1}{Y}$  provided  $Y + Y^{-1} = o(D^{1/6-\epsilon})$  as  $D \rightarrow \infty$ , while Theorem 1.3 sharpens this to a slowly varying rectangle  $[a, b] \times [c, d] \subset [0, 1] \times (0, \infty]$ , by an application of Weyl's criterion. Theorem 1.1 then follows from these two by approximating a given measurable set  $B$  with rectangles.

To place Theorem 1.1 in context, the most readily available comparison is to the Heegner points attached to the 2-part of the class group. For  $d \equiv 2 \pmod{4}$ ,  $H_2(-d)$  has  $\frac{\tau(d)}{2}$  ideal classes indexed by the divisors of  $d$ ; for  $d_1 d_2 = d$ ,  $d_1 < \sqrt{d}$ ,

$$\mathfrak{a}_{d_1} = \prod_{p|d_1, p^2|(p)} p, \quad \mathfrak{a}_{d_1} = [d_1, \sqrt{-d}], \quad \mathfrak{a}_{d_1} \sim \mathfrak{a}_{d_2} \in H_2(-d).$$

The Heegner points of  $H_2(-d)^*$  in the fundamental domain  $\mathcal{F}$  for  $SL_2(\mathbb{Z}) \backslash \mathbb{H}$  are thus the collection of points

$$\left\{ i\sqrt{\frac{d_2}{d_1}} \right\}_{\substack{d_1 d_2 = d \\ 1 < d_1 < \sqrt{d}}}.$$

One may check that as  $0 < d < D$  varies this set equidistributes on the imaginary axis between 1 and  $D^{\frac{1}{2}-\epsilon}$  with respect to the one-dimensional hyperbolic metric.

In the parallel real quadratic setting Heegner cycles (closed geodesics) replace Heegner points, and Sarnak [13] has considered several average equidistribution problems for even ordered elements in the class group. In [13] it is shown that the geodesics associated to ambiguous quadratic forms do not equidistribute in  $\Gamma \backslash \mathbb{H}$  when ordered by arc-length, but he makes progress toward showing that geodesics of reciprocal forms do equidistribute with the same ordering, and he conjectures that the geodesics of both families of forms equidistribute when ordered by discriminant. It would be a natural extension of our work to consider average equidistribution of the Heegner cycles associated to the 3-part for real quadratic fields, since both the Davenport-Heilbronn theorem and Duke's result continue to hold in this context, but we have not yet succeeded in doing this.

We have recently learned of a third parallel to our result concerning the shape of the maximal order in a cubic field, considered as a three dimensional lattice in  $\mathbb{R}^3$ . In [16] David Terr proves that when either real or imaginary cubic fields are ordered by discriminant the two dimensional lattice which is the projection of the field's maximal order in the plane perpendicular to the direction of 1 equidistributes in  $SL_2(\mathbb{Z}) \backslash \mathbb{H}$ , this

space being identified with the space of 2-dimensional lattices modulo dilation and rotation. Terr's result is especially connected to our work because Hasse [8] proved a correspondance between the 3-part of the class group of a real quadratic field and nowhere-totally-ramified cubic fields of the same discriminant. It would be interesting to see whether a precise statement of Hasse's correspondance at the level of lattices together with a refinement of Terr's method can be made to give an alternate proof of Theorem 1.1.

Returning to the imaginary quadratic setting, the distinction between the equidistribution for Heegner points of  $H_2(-d)^*$  on the imaginary axis and  $H_3(-d)^*$  throughout  $\mathcal{F}$  is not surprising, given that elements of order two are special in the class group of a quadratic field extension. One might reasonably hope, however, that the analogue of Theorem 1.1 holds when  $H_3(-d)^*$  is replaced by  $H_k(-d)^*$  for any odd  $k$ , with  $H_k(-d)^*$  the collection of ideal classes of order  $k$ . In this direction we are able to prove the following weaker version of Theorem 1.2.

**Theorem 1.4.** *Let  $k > 1$  be odd and take  $\phi, \psi$  non-negative functions in  $C^\infty(\mathbb{R}^+)$ ,  $\phi$  compactly supported,  $\psi$  Schwartz class with Mellin transform  $\hat{\psi}$  entire except for possibly a simple pole at zero. Define*

$$S_k(D, Y; \phi, \psi) = \sum_{\substack{d \equiv 2 \pmod{4} \\ \text{squarefree}}} \phi\left(\frac{d}{D}\right) \sum_{\mathfrak{a} \in H_k(-d)^* \text{ primitive}} \psi\left(\frac{\mathfrak{I}(z_{\mathfrak{a}})^{-1}}{Y}\right).$$

For  $D^{-\frac{1}{2} + \frac{1}{k}} < Y < D^{O(1)}$  we have

$$\begin{aligned} S(D, Y; \phi, \psi) &= \frac{6}{\pi^3} \hat{\phi}(1) \hat{\psi}(1) Y D + C_{1,k} \hat{\phi}\left(\frac{1}{2} + \frac{1}{k}\right) \text{Res}_{z=0} \hat{\psi}(z) D^{\frac{1}{2} + \frac{1}{k}} \\ &\quad + o(D^{\frac{1}{2} + \frac{1}{k}}) + O(Y^{1+\frac{k}{4}} D^{\frac{1}{2} + \frac{k}{8} + \epsilon}) + O((1+Y) Y^{\frac{k}{2}} D^{\frac{k}{4} + \epsilon}) + O(Y^{\frac{1}{2}} D^{\frac{3}{4} + \epsilon}); \end{aligned}$$

$$C_{1,k} = \frac{1}{6k} \frac{\zeta(1 - \frac{2}{k})}{\zeta(2)} \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{1}{2} - \frac{1}{k})}{\Gamma(1 - \frac{1}{k})} [1 - 2^{\frac{1}{k}} + 2^{1 - \frac{1}{k}}] \prod_{p \text{ odd}} \left[ 1 + \frac{1}{p+1} \left( \frac{1}{p^{\frac{1}{k}}} - \frac{1}{p^{1 - \frac{2}{k}}} - \frac{1}{p^{1 - \frac{1}{k}}} - \frac{1}{p} \right) \right].$$

For  $k > 3$  this gives an asymptotic in the range  $D^{\frac{1}{k} - \frac{1}{2}} < Y < D^{\frac{1}{k-2} - \frac{1}{2} - \epsilon}$ , and hence establishes a constant vertical density of  $H_k(-d)^*$  Heegner points in a band high in the cusp. This should be compared to the case  $k = 3$  in Theorem 1.2 where we obtained an asymptotic for

$$D^{-\frac{1}{6} + \epsilon} < \mathfrak{I}(z) < D^{\frac{1}{6} - \epsilon}.$$

Whereas the band for  $k = 3$  expands to cover the strip  $\Gamma_\infty \backslash \mathbb{H}$  allowing us to prove the equidistribution result in Theorem 1.1, for  $k > 3$  the corresponding region vanishes into the cusp and we do not know if equidistribution holds. For  $Y < D^{-\frac{1}{2} + \frac{1}{k} + \min(\frac{1}{k+4}, \frac{2}{k^2}) - \epsilon}$  we obtain in addition a negative<sup>1</sup> secondary main term. Unlike the primary main term, this secondary term does not vary with  $Y$  and instead of equidistribution, it records the fact that for  $0 < d < D$ ,  $H_k(-d)^*$  has no associated Heegner points with  $\mathfrak{I}(z_{\mathfrak{a}}) > D^{\frac{1}{2} - \frac{1}{k}}$  because  $\mathfrak{a} \in H_k(-d)^*$  implies  $\mathfrak{a}^k$  is principal so that

$$(N\mathfrak{a})^k = N(\mathfrak{a}^k) \geq d \quad \Rightarrow \quad \mathfrak{I}(z_{\mathfrak{a}}) = \frac{\sqrt{d}}{N\mathfrak{a}} \leq d^{\frac{1}{2} - \frac{1}{k}}.$$

This suggests a negative secondary term in the mean size of  $H_k(-d)^*$ , as we now describe.

Average equidistribution of  $H_k(-d)^*$  points and the mean size of  $H_k(-d)^*$  are connected by the relation

$$\sum_{d \leq D}^* |H_k(-d)^*| = \sum_{d \leq D}^* \sum_{\mathfrak{a} \in H_k(-d)^* \text{ primitive}} \mathbf{1}_{\mathcal{F}}(z_{\mathfrak{a}}),$$

which holds because each ideal class in  $H(-d)$  has exactly one point in a fundamental domain  $\mathcal{F}$ . In particular, for  $k = 3$  the Davenport-Heilbronn asymptotic is a consequence of Theorem 1.1, with a minor

<sup>1</sup>Negative because  $\zeta(1 - \frac{2}{k}) < 0$ .

modification to account for the fact that  $\mathcal{F}$  is not compact<sup>2</sup>. What is perhaps at first sight surprising is that calculating the mean size of  $H_k(-d)^*$  is essentially equivalent to establishing equidistribution of just the imaginary part of the Heegner points near  $\Im(z) \doteq 1$ . Precisely, for a special choice of the test function  $\psi$  we can write the aggregate size of  $H_k(-d)^*$  exactly as  $\mathcal{S}(D, 1; \phi, \psi)$ .

**Theorem 1.5.** *Let  $\Psi_0(y) = (2\pi y - 1)e^{-\pi y}$  and  $\Psi(y) = \sum_{m=1}^{\infty} \Psi_0(m^2 y)$ . For  $\phi \in C_c^\infty(\mathbb{R}^+)$  we have*

$$\sum_d^* \phi\left(\frac{d}{D}\right) |H_k(-d)^*| = 2 \sum_d^* \phi\left(\frac{d}{D}\right) \sum_{a \in H_k(-d)^* \text{ primitive}} \Psi\left(\frac{1}{\Im z_a}\right).$$

Since the Mellin Transform of  $\Psi$  is

$$\hat{\Psi}(s) = (2s - 1)\pi^{-s}\Gamma(s)\zeta(2s),$$

$\Psi$  satisfies the conditions of Theorem 1.4 with  $\hat{\Psi}(1) = \frac{\pi}{6}$  and  $\text{Res}_{s=0} \hat{\Psi}(s) = \frac{1}{2}$ . If we assume that equidistribution holds for all odd  $k \geq 3$  and with sufficient strength so that the absence of Heegner points above  $\Im(z) = D^{\frac{1}{2} - \frac{1}{k}}$  is not obscured, then we are led to the following conjecture for the mean size of  $H_k(-d)^*$ .

**Conjecture 1.1.** *Let  $\phi \in C_c^\infty(\mathbb{R}^+)$ . The smoothed aggregate size of  $H_k(-d)^*$  is given by*

$$(1) \quad \sum_d^* \phi\left(\frac{d}{D}\right) |H_k(-d)^*| = \frac{2}{\pi^2} \hat{\phi}(1) D + C_{1,k} \hat{\phi}\left(\frac{1}{2} + \frac{1}{k}\right) D^{\frac{1}{2} + \frac{1}{k}} + o(D^{\frac{1}{2} + \frac{1}{k}})$$

with  $C_{1,k}$  the constant from Theorem 1.4.

Thus we expect that the mean size of  $H_k(-d)^*$  is described by the equidistribution of its Heegner points in  $\mathcal{F}$ , with a correction for the fact that there are no ideals of order  $k$  with small norm.

To place Conjecture 1.1 in relation to previous conjectures, observe that

$$\sum_{d \equiv 2 \pmod{4}} \phi\left(\frac{d}{D}\right) \sim \frac{2}{\pi^2} D$$

so that leading term in (1) coincides with the Cohen-Lenstra heuristic in asserting that for each fixed odd  $k$  there is on average one element in  $H(-d)$  of order  $k$ . As regards the secondary term, for  $k = 3$  Roberts [12], building on work of Datskovsky and Wright ([5], [19], [4]), has proposed and given substantial numerical evidence for a negative secondary term  $C_{\text{Roberts}} D^{\frac{5}{6}}$  in the count of cubic fields with discriminant less than  $D$ . This leads to a modified secondary term  $C'_{\text{Roberts}} D^{\frac{5}{6}}$  in  $\sum_{d \leq D}^* |H_3(-d)^*|$  because Hasse's correspondence gives that  $|H_3(-d)^*|$  is equal to twice the number of nowhere-totally-ramified cubic fields of discriminant  $d$ , with conjugate fields being identified.<sup>3</sup> If in (1) we take for  $\phi$  the function  $\chi_{[0,1]}$  so that  $\hat{\phi}(s) = \frac{1}{s}$ , the secondary term of (1) matches Roberts' modified conjecture. For  $k > 3$  Venkatesh [18] has suggested a negative secondary term of shape  $D^{\frac{1}{2} + \frac{1}{k}}$  based upon experimental evidence, but our conjectured constant is new.

From our point of view, Conjecture 1.1 appears to be far from reach except for the secondary term for  $k = 3$  and possibly the main term for  $k = 5$ . For  $k = 3$ , Bhargava, Shankar and Tsimerman [2] and separately Thorne [17] have recently made a major breakthrough, establishing Roberts' secondary main term for the number of cubic fields of bounded discriminant. At the time of this writing neither group had announced the corresponding result for the mean size of  $H_3(-d)^*$ , but there is every reason to hope that this conjecture will be established by a modification of their methods. For the  $k = 3$  case, our method yields the following smoothed version of the Davenport-Heilbronn theorem.

**Theorem 1.6** (Smoothed Davenport-Heilbronn). *Let  $\phi \in C_c^\infty(\mathbb{R}^+)$ . We have*

$$\sum_d^* \phi\left(\frac{d}{D}\right) |H_3(-d)^*| = \frac{2}{\pi} \hat{\phi}(1) D + O(D^{\frac{7}{8} + \epsilon}).$$

<sup>2</sup>One could write  $\mathcal{F} = \mathcal{F}_1 \sqcup \mathcal{F}_2$  with  $\mathcal{F}_1 = \mathcal{F} \cap \{z : \Im(z) \leq 1\}$  and apply Theorem 1.2 to  $\mathcal{F}_2$ .

<sup>3</sup>This fact was used by Davenport and Heilbronn in [6] to deduce the mean size of  $H_3(-d)$  from their count for the number of cubic extensions of  $\mathbb{Q}$  of a bounded discriminant.

The error term of  $O(D^{\frac{7}{8}+\epsilon})$  matches the best recorded error for the mean size of  $H_3(-d)^*$  due to Belabas, Bhargava and Pomerance in [1], although one should note that the result in [1] does not require smoothing and obtains the mean size of  $H_3(-d)^*$  by a rather different method, as a consequence of a count for cubic fields through Hasse's relation. For  $k > 3$  we cannot get an asymptotic, but following Soundararajan in [14] we obtain upper and lower bounds by applying Theorem 1.4 with a function  $\psi(z) = \psi(y)$  chosen to either majorize or minorize  $1_{\mathcal{F}}(z)$ .

**Theorem 1.7.** *Let  $k > 3$  be odd. We have*

$$D^{\frac{1}{2} + \frac{1}{k-2} - \epsilon} \ll \sum_{d \leq D}^* |H_k(-d)^*| \ll D^{\frac{k}{4} + \epsilon}.$$

The upper bound here is non-trivial only for  $k = 5$ , which is the case that was stated in [14].

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## 2. BACKGROUND CONCERNING HEEGNER POINTS AND IMAGINARY QUADRATIC FIELDS

For completeness we review the part of the theory of Heegner points and imaginary quadratic fields that we need for our results. Most of this material can be found in Chapter 22 of [11].

Throughout we will assume that  $d > 0$  is squarefree,  $d \equiv 2 \pmod{4}$  so that the imaginary quadratic field  $K = \mathbb{Q}(\sqrt{-d})$  has ring of integers  $\mathcal{O} = \mathbb{Z}[\sqrt{-d}]$ . We have unique factorization of ideals in  $\mathcal{O}$  with the primes of  $\mathcal{O}$  derived from those in  $\mathbb{Z}$  according to quadratic character<sup>4</sup> of  $-d \pmod{p}$

$$p\mathcal{O} = \begin{cases} p^2 & p|d \\ p\bar{p} & \left(\frac{-d}{p}\right) = 1 \\ p\mathcal{O} & \left(\frac{-d}{p}\right) = -1 \end{cases}.$$

We say that  $p$  either ramifies, splits, or remains inert. The different is the product of primes containing  $d$ ,

$$\mathfrak{d} = \prod_{p|(d)} p.$$

In the introduction we described the primitive ideals as those ideals which do not contain an integer prime factor. The following are two other useful characterizations.

**Proposition 2.1.** *An ideal  $\mathfrak{a}$  of  $\mathcal{O}$  is primitive if and only if it factors as  $\mathfrak{a} = \mathfrak{l}\mathfrak{b}$  with  $\mathfrak{l}|\mathfrak{d}$ ,  $(\mathfrak{b}, \mathfrak{d}) = (1)$  and  $(\mathfrak{b}, \bar{\mathfrak{b}}) = (1)$ . In particular,  $\mathfrak{b}$  contains only primes  $p$  dividing split primes, with at most one of  $p, \bar{p}$  appearing.*

The norm  $N\mathfrak{a}$  of an ideal  $\mathfrak{a}$  is the number of distinct residue classes in  $\mathcal{O}/\mathfrak{a}$ .

**Proposition 2.2.** *An ideal  $\mathfrak{a}$  of  $\mathcal{O}$  is primitive if and only if the set  $\{1, 2, \dots, N\mathfrak{a}\}$  forms a complete set of residues for  $\mathcal{O}/\mathfrak{a}$ .*

The second characterization makes it clear that a primitive ideal  $\mathfrak{a}$  may be expressed uniquely as a  $\mathbb{Z}$ -module in the form  $\mathfrak{a} = [N\mathfrak{a}, b + \sqrt{-d}]$  with  $\frac{N\mathfrak{a}}{2} < b \leq \frac{N\mathfrak{a}}{2}$ ; the choice of  $b \equiv -\sqrt{-d} \pmod{\mathfrak{a}}$  is forced.

The Heegner point associated to the primitive ideal  $\mathfrak{a}$  is the point in the upper half plane  $\mathbb{H}$

$$z_{\mathfrak{a}} = \frac{b + \sqrt{-d}}{N\mathfrak{a}}.$$

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<sup>4</sup> $\left(\frac{\cdot}{p}\right)$  is the Legendre symbol.

Since this point is the ratio of two vectors generating the lattice  $\mathfrak{a}$ , two lattices  $\mathfrak{a}$  and  $\mathfrak{a}'$  of the same shape have Heegner points related by a fractional linear transformation

$$z_{\mathfrak{a}} = \gamma \cdot z_{\mathfrak{a}'}, \quad \gamma \in \Gamma = \mathrm{PSL}_2(\mathbb{Z})$$

corresponding to an integer change of basis. In particular, the Heegner point of the ideal class  $[\mathfrak{a}]$  of  $\mathfrak{a}$  is well defined in  $\Gamma \backslash \mathbb{H}$ . We take this point to lie in the standard fundamental domain  $\mathcal{F}$  for  $\Gamma \backslash \mathbb{H}$ . Let

$$\Gamma_{\infty} = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$$

be the subgroup of  $\Gamma$  stabilizing  $\infty$ . The strip  $\{z \in \mathbb{H} : \frac{-1}{2} < \Re(z) \leq 1\}$  is a fundamental domain for  $\Gamma_{\infty} \backslash \mathbb{H}$ . The following proposition is basic to our arguments

**Proposition 2.3.** *Let  $d \equiv 2 \pmod{4}$  and fix a class  $[\mathfrak{a}] \in H(-d)^*$ . The mapping  $\mathfrak{a} \mapsto z_{\mathfrak{a}}$  establishes a bijection between primitive ideals of class  $[\mathfrak{a}]$  in  $\mathcal{O}$  and the set*

$$(\Gamma_{\infty} \backslash \Gamma) \cdot z_{[\mathfrak{a}]} = \left\{ \gamma \cdot z_{[\mathfrak{a}]} : \gamma \in \Gamma, \frac{-1}{2} < \Re(\gamma \cdot z_{[\mathfrak{a}]}) \leq \frac{1}{2} \right\} \subset \Gamma_{\infty} \backslash \mathbb{H}.$$

*Proof.* Since  $\frac{-N\mathfrak{a}}{2} \leq \mathfrak{b} \leq \frac{N\mathfrak{a}}{2}$  we have  $z_{\mathfrak{a}} \in \Gamma_{\infty} \backslash \mathbb{H}$ , and  $z_{\mathfrak{a}}$  is related to  $z_{[\mathfrak{a}]}$  by some  $\gamma \in \Gamma$ . Moreover, since  $d$  is fixed we can recover both  $\mathfrak{b}$  and  $N\mathfrak{a}$  from the point  $z_{\mathfrak{a}}$  so that map  $\mathfrak{a} \mapsto z_{\mathfrak{a}}$  is an injection into the set  $(\Gamma_{\infty} \backslash \Gamma) \cdot z_{[\mathfrak{a}]}$ .

To prove that the map is onto, take any  $\mathfrak{a}$  with associated point  $z_{\mathfrak{a}}$  and let  $z'$  be another point of  $(\Gamma_{\infty} \backslash \Gamma) \cdot z_{[\mathfrak{a}]}$ . Choose  $\begin{pmatrix} k & l \\ m & n \end{pmatrix} = \gamma \in \Gamma$  with  $\gamma \cdot z_{\mathfrak{a}} = z'$ . The ideal

$$\begin{aligned} \mathfrak{b} &= (lN\mathfrak{a} + k(b - \sqrt{-d})) \mathfrak{a} \\ &= (lN\mathfrak{a} + k(b - \sqrt{-d})) [lN\mathfrak{a} + k(b + \sqrt{-d}), mN\mathfrak{a} + n(b + \sqrt{-d})] \\ &= [(lN\mathfrak{a} + kb)^2 + dk^2, ((lm(N\mathfrak{a})^2 + (ln + km)bN\mathfrak{a} + kn(b^2 + d)) + (n(lN\mathfrak{a} + kb) - k(mN\mathfrak{a} + nb))\sqrt{-d}] \end{aligned}$$

may not be primitive, but dividing through by an integer factor produces a primitive ideal  $\mathfrak{c}$  without changing the ratio between the  $\mathbb{Z}$ -module generators, which is  $z'$ . It also preserves the fact that the first generator above is an integer, which then must be the norm of the resulting ideal, so that  $z' = z_{\mathfrak{c}}$ . This establishes the surjection.  $\square$

We will also make several applications of the following geometric estimate.

**Proposition 2.4.** *For any  $z \in \mathbb{H}$ ,  $|\{\gamma \in \Gamma_{\infty} \backslash \Gamma : \Im(\gamma z) \geq Y^{-1}\}| \ll 1 + Y$ .*

*Proof.* See [10] Lemma 2.11.  $\square$

From Propositions 2.3 and 2.4 we obtain a basic upper bound for the number of ideals in a given class having bounded norm.

**Corollary 2.1.** *Fix  $d$  and an ideal class  $[\mathfrak{a}] \in H(-d)$ . We have the bound*

$$|\{\mathfrak{b} : [\mathfrak{b}] = [\mathfrak{a}], N\mathfrak{b} \leq Y\sqrt{d}\}| \ll 1 + Y.$$

*Proof.* The condition  $N\mathfrak{b} \leq Y\sqrt{d}$  is equivalent to  $\Im(z_{\mathfrak{b}}) \geq Y^{-1}$ . Since  $z_{\mathfrak{b}} = \gamma \cdot z_{[\mathfrak{a}]}$  for some  $\gamma \in \Gamma_{\infty} \backslash \Gamma$  the result is a consequence of Proposition 2.4.  $\square$

### 3. PARAMETRIZATION OF NON-TRIVIAL PRIMITIVE IDEALS IN $H_k(-d)$

Our proofs follow the method of Soundararajan [14] in which he counts solutions to a certain equation, which parametrize non-trivial primitive ideals of  $H_k(-d)$ . In the case  $d \equiv 2 \pmod{4}$  the bijection is as follows.

**Proposition 3.1.** *Let  $d \equiv 2 \pmod{4}$  be squarefree and  $k \geq 3$  be odd. The set*

$$\{(l, m, n, t) \in (\mathbb{Z}^+)^4 : lm^k = l^2n^2 + t^2d, l|d, (m, ntd) = 1\}$$

*is in bijection with primitive ideal pairs  $(\mathfrak{a}, \bar{\mathfrak{a}})$  with  $\mathfrak{a} \neq (1)$  and  $\mathfrak{a}^k$  principal in  $\mathbb{Q}(\sqrt{-d})$ . Explicitly, the ideal  $\mathfrak{a}$  is given as a  $\mathbb{Z}$ -module by*

$$\mathfrak{a} = [lm, lnt^{-1} + \sqrt{-d}]$$

*where  $N\mathfrak{a} = lm$  and  $t^{-1}$  is the inverse of  $t$  modulo  $m$ .*

*Proof.* Take  $\mathfrak{a} \neq (1)$  primitive with  $\mathfrak{a}^k$  principal and write  $\mathfrak{a} = l\mathfrak{b}$  where  $l|d$  and  $(b, d) = (1)$ . We have  $b \neq (1)$  since otherwise  $\mathfrak{a} = l \Rightarrow [l]^k = [l] = [1]$  which, together with primitivity of  $\mathfrak{a}$ , would force  $l = (1)$ . Now

$$(2) \quad \mathfrak{a}^k l^{-(k-1)} = (x + t\sqrt{-d})$$

is principal. It is also primitive since  $(x + t\sqrt{-d}) = l\mathfrak{b}^k$  and  $(b, d) = (1)$ ,  $(b, d) = 1$ . Let  $m = Nb$ ,  $l = Nl$  and take norms in eqn. (2) to obtain  $lm^k = x^2 + t^2d$ . Here  $l|x$  so writing  $x = ln$ ,  $m^k = ln^2 + t^2\bar{l}$  where  $\bar{l} = d$ . Now primitivity of the ideal  $(ln + t\sqrt{-d})$  implies  $(t, ln) = 1$ . We claim  $(m, ntd) = 1$ . Indeed, suppose  $p$  is a prime dividing  $(m, d)$ , say  $p|l$ . Then  $p|t$  so  $p^2|ln^2$  which forces  $p|n$ , a contradiction. If instead  $p|nt$  say  $p|n$ . Then  $p^2|t^2\bar{l}$  so  $p|t$ , again in contradiction. The remaining cases are symmetric to the previous two. Finally, primitivity of  $(ln + t\sqrt{-d})$  implies  $n, t \neq 0$ . We may fix  $n > 0$  by multiplying  $\mathfrak{a}$  by a unit; the choice of sign for  $t$  is determined by a choice between the ideals  $\mathfrak{a}$  and  $\bar{\mathfrak{a}}$ .

Now suppose we begin with a solution  $(l, m, n, t)$  to  $lm^k = l^2n^2 + t^2d$  with  $l|d$ ,  $m, n, t > 0$ ,  $(m, ntd) = 1$ . Write  $(ln + t\sqrt{-d}) = l\mathfrak{c}$  where  $l|d$  and  $(c, d) = 1$ . Then  $(l)(m^k) = l^2c\bar{c}$  and  $(m, d) = 1$  implies  $l^2 = (l)$  and  $c\bar{c} = (m^k)$ . Moreover,  $c$  is primitive since it divides  $(ln + t\sqrt{-d})$ , and  $c$  is prime to  $d$  so  $(c, \bar{c}) = 1$ , and hence there exists  $b$  with  $c = b^k$ ,  $\bar{c} = \bar{b}^k$ . Note that  $(b, d) = 1$  and  $b$  is primitive. Then letting  $\mathfrak{a} = l\mathfrak{b}$ ,  $\bar{\mathfrak{a}} = l\bar{\mathfrak{b}}$  we get that  $(\mathfrak{a}, \bar{\mathfrak{a}}) \neq ((1), (1))$  is a pair of primitive ideals satisfying  $\mathfrak{a}^k = (l)(ln + t\sqrt{-d})$  is principal. Since there were no choices in determining the pair  $(\mathfrak{a}, \bar{\mathfrak{a}})$ , this completes the bijection.

Taking  $\mathfrak{a}$  to be the ideal in the pair  $(\mathfrak{a}, \bar{\mathfrak{a}})$  that corresponds to  $n, t > 0$ , we now specify  $\mathfrak{a}$  in terms of  $l, m, n, t$ . Since  $\mathfrak{a}$  is primitive,  $\mathfrak{a} = [Na, b + \sqrt{-d}]$  as a  $\mathbb{Z}$ -module, where  $b$  is determined modulo  $Na$ . From the above bijection,  $Na = lm$ , so it remains to determine  $b \pmod{lm}$ . Now

$$\mathfrak{a}^2 = (l)b^2 = [l^2m^2, lmb + lmt\sqrt{-d}, b^2 - d + 2b\sqrt{-d}].$$

For the right side to be divisible by  $l$ , we must have  $l|b^2 - d$  so  $l|b^2 \Rightarrow l|b$  so write  $b = lb'$ . Hence

$$\mathfrak{a}^k l^{-(k-1)} = l\mathfrak{b}^k = \left[ l^{\frac{k+1}{2}} m^k, l^{\frac{k+1}{2}} m^{k-1} b' + l^{\frac{k-1}{2}} m^{k-1} \sqrt{-d}, *, \dots, * \right] = (ln + t\sqrt{-d}).$$

A necessary condition for this last equality to hold is that  $l^{\frac{k+1}{2}} m^{k-1} b' + l^{\frac{k-1}{2}} m^{k-1} \sqrt{-d} \in (ln + t\sqrt{-d})$  so for some integers  $x, y$ ,

$$\begin{aligned} l^{\frac{k+1}{2}} m^{k-1} b' + l^{\frac{k-1}{2}} m^{k-1} \sqrt{-d} &= (ln + t\sqrt{-d})(x + y\sqrt{-d}) \\ &\Rightarrow l^{\frac{k+1}{2}} m^k b' + l^{\frac{k-1}{2}} m^k \sqrt{-d} = (ln + t\sqrt{-d})(mx + my\sqrt{-d}). \end{aligned}$$

But  $lm^k = (ln + t\sqrt{-d})(ln - t\sqrt{-d})$  so we conclude

$$\begin{aligned} (ln - t\sqrt{-d})(l^{\frac{k-1}{2}} m^{k-1} b' + l^{\frac{k-3}{2}} m^{k-1} \sqrt{-d}) &= (l^{\frac{k+1}{2}} m^k b' + l^{\frac{k-3}{2}} m^k \sqrt{-d}) + (l^{\frac{k-1}{2}} m^k n - l^{\frac{k-1}{2}} m^k t\sqrt{-d}) \\ &= mx + my\sqrt{-d}. \end{aligned}$$

Hence

$$l^{\frac{k-1}{2}} m^{k-1} n \equiv tl^{\frac{k-1}{2}} m^{k-1} b' \pmod{m} \quad \Rightarrow \quad b' \equiv t^{-1} n \pmod{m}$$

and  $b = lb' \equiv lnt^{-1} \pmod{lm}$  as claimed.  $\square$

#### 4. VERTICAL EQUIDISTRIBUTION OF HEEGNER POINTS ASSOCIATED TO THE 3-PART OF THE CLASS GROUP

We use the parametrization of the previous section to prove Theorem 1.2. Recall that we defined

$$(3) \quad \mathcal{S}(D, Y) = \sum_{\substack{0 > -d > -D \\ d \equiv 2 \pmod{4} \\ \text{squarefree}}} \sum_{\substack{\mathfrak{a} \text{ primitive} \\ [\mathfrak{a}] \in H_3(-d)^* \\ \mathfrak{I}(z_{\mathfrak{a}}) > \frac{1}{Y}}} 1$$

and that the condition  $\mathfrak{I}(z_{\mathfrak{a}}) > \frac{1}{Y}$  is equivalent to  $N\mathfrak{a} \leq Y\sqrt{d}$  since  $z_{\mathfrak{a}} = \frac{b + \sqrt{-d}}{N\mathfrak{a}}$ .

The set parametrized in Proposition 3.1 differs from the ideals we wish to count in  $\mathcal{S}(D, Y)$  because it includes the primitive principal ideals other than (1). This is not a great difficulty, however, since the number of these satisfying  $d < D, N\mathfrak{a} < Y\sqrt{d}$  is few in a wide range of  $Y$ .

**Lemma 4.1.** *We have the bound*

$$\sum_{d \leq D}^* \sum_{\substack{(1) \neq \mathfrak{a} \text{ primitive} \\ [\mathfrak{a}] = [(1)] \in H(-d) \\ N\mathfrak{a} \leq Y\sqrt{d}}} 1 = O(Y^3).$$

*Proof.* We may assume  $Y > 1$  since otherwise the sum is empty. If  $\mathfrak{a} \neq (1)$  is a primitive principal ideal of  $\mathbb{Q}(\sqrt{-d})$  then  $\mathfrak{a} = (u + v\sqrt{-d})$  with  $v \neq 0$ , so  $N\mathfrak{a} \geq d$ . It follows that such  $\mathfrak{a}$  exist only if  $d \leq Y^2$ , and given  $d$ , the number of  $\mathfrak{a}$  is  $\ll Y$  by Corollary 2.1.  $\square$

We may thus estimate

$$\mathcal{S}(D, Y) + O(Y^3) = 2 \sum_{d \leq D}^* \# \left\{ (l, m, n, t) \in (\mathbb{Z}^+)^4 : lm^3 = l^2n^2 + t^2d, l|d, (m, ntd) = 1, lm \leq Y\sqrt{d} \right\}$$

Writing

$$S(d, Y) = \left\{ (l, m, n, t) \in (\mathbb{Z}^+)^4 : lm^3 = l^2n^2 + t^2d, l \text{ squarefree}, l|d, (m, ntd) = 1, lm \leq Y\sqrt{d} \right\}$$

and sieving for squarefree discriminants we obtain

$$\begin{aligned} \mathcal{S}(D, Y) + O(Y^3) &= 2 \sum_{\substack{d \leq D \\ d \equiv 2 \pmod{4}}} \sum_{s^2|d} \mu(s) |S(d, Y)| \\ &= 2 \sum_{\substack{d \leq D \\ d \equiv 2 \pmod{4}}} \left\{ \sum_{s^2|d, s < Z} + \sum_{s^2|d, s \geq Z} \right\} \mu(s) |S(d, Y)| = 2S_1 + 2S_2, \end{aligned}$$

with  $Z \geq 1$  a parameter to be chosen. Also, by ignoring the condition  $d$  squarefree and counting principal ideals, we have an upper bound

$$(4) \quad \mathcal{S}(D, Y) \leq 2 \sum_{d \leq D} |S(d, Y)| = 2S_3.$$

**Proposition 4.1.** *For  $Y > D^{-1/6}$ ,*

$$\begin{aligned} S_1 &= \frac{3YD}{\pi^3} + \frac{1}{2} C_{5/6} D^{5/6} \\ &\quad + O \left( Y^{5/2} Z (\log Z) D^{3/4} (\log D)^2 + YDZ^{-1} + D^{5/6} \exp \left( -c (\log Y^{3/2} D^{1/4})^{1/3} \right) \right) \end{aligned}$$

with  $C_{5/6}$  the constant from Theorem 1.2.

**Proposition 4.2.** *For  $Y > D^{-1/6}$ ,  $S_2 = O\left(\frac{YD^{1+\epsilon}}{Z}\right)$ .*

Our proof of Proposition 4.2 requires the following upper bound.



**Proposition 4.3.** *Uniformly in  $Y > 0$ , and for any fixed  $\epsilon > 0$ ,  $S(D, Y) \ll_\epsilon Y D^{1+\epsilon}$ .*

Taking  $Z = Y^{-3/4} D^{1/8}$  in Propositions 4.1 and 4.2 we obtain Theorem 1.2.

**4.1. The main term  $S_1$ .** Our first goal is rearrange the sum in  $S_1$  so that it is in a form that we can estimate. To this end, we note that by solving for  $d$  in terms of  $l, m, n, t$ ,

$$d = \frac{lm^3 - l^2 n^2}{t^2}$$

and so the conditions  $d \leq D$  and  $lm \leq Y\sqrt{d}$  are equivalent to  $n^2 \geq \frac{m^3}{l} - \frac{t^2 D}{l^2}$ , and  $n^2 \leq \frac{m^3}{l} - \frac{t^2 m^2}{Y^2}$ . The second of these implies

$$l^2 t^2 \leq Y^2 lm \leq Y^3 \sqrt{d}.$$

Hence we may write

$$(5) \quad S_1 = \sum_{l \leq Y^{3/2} D^{1/4}} \sum_{\substack{t \leq \frac{Y^{3/2} D^{1/4}}{l} \\ (t, l) = 1}} \sum_{\substack{m \leq \frac{Y\sqrt{D}}{l} \\ (m, 2lt) = 1}} \sum_{\substack{(n, m) = 1 \\ \frac{m^3}{l} - \frac{t^2 D}{l^2} \leq n^2 \leq \frac{m^3}{l} - \frac{t^2 m^2}{Y^2} \\ lm^3 - l^2 n^2 \equiv 2t^2 \pmod{4t^2}}} \sum_{\substack{s^2 | \frac{lm^3 - l^2 n^2}{t^2} \\ s < Z}} \mu(s)$$

By Möbius inversion and rearranging order of summation, the last two sums may be written

$$\sum_{\substack{(n, m) = 1 \\ \frac{m^3}{l} - \frac{t^2 D}{l^2} \leq n^2 \leq \frac{m^3}{l} - \frac{t^2 m^2}{Y^2} \\ lm^3 - l^2 n^2 \equiv 2t^2 \pmod{4t^2}}} \sum_{\substack{s^2 | \frac{lm^3 - l^2 n^2}{t^2} \\ s < Z}} \mu(s) = \sum_{d | m} \mu(d) \sum_{\substack{s < Z \\ (s, 2lm) = 1}} \mu(s) \sum_{\substack{\frac{m^3}{l} - \frac{t^2 D}{l^2} \leq d^2 n^2 \leq \frac{m^3}{l} - \frac{t^2 m^2}{Y^2} \\ lm^3 - l^2 d^2 n^2 \equiv 2s^2 t^2 \pmod{4s^2 t^2}}} 1.$$

This has a desirable form because the sum over  $n$  is long (of length  $D^{\frac{3}{4}}$ ) compared to the sums over  $m$  ( $D^{\frac{1}{2}}$ ) and  $t$  ( $D^{\frac{1}{4}}$ ).

We can now estimate the inner sum over  $n$  by breaking it into blocks of length  $4s^2 t^2$ . Writing

$$\rho_{m, l, d}(r) = \# \{n \pmod{4r} : lm^3 - l^2 d^2 n^2 \equiv 2r \pmod{4r}\},$$

each such block contributes  $\rho_{m, l, d}(s^2 t^2)$  to the sum, so that<sup>5</sup>

$$(6) \quad S_1 = \sum_{\substack{lt \leq Y^{3/2} D^{1/4} \\ (t, l) = 1 \\ l \text{ squarefree}}} \sum_{\substack{m \leq \frac{Y\sqrt{D}}{l} \\ (m, 2lt) = 1}} \sum_{d | m} \mu(d) \sum_{\substack{s < Z \\ (s, 2lm) = 1}} \mu(s) \rho_{m, l, d}(s^2 t^2) \left[ \frac{1}{4s^2 t^2 d} \left( \sqrt{\frac{m^3}{l} - \frac{l^2 m^2}{Y^2}} - \sqrt{\frac{m^3}{l} - \frac{t^2 D}{l^2}} \right) + O(1) \right]$$

**Lemma 4.2.** *Say  $2^a || r$  and  $d$  is odd.*

$$\rho_{m, l, d}(r) = \sigma_{m, l}(2^a) \prod_{p | r \text{ odd}} \left( 1 + \left( \frac{lm}{p} \right) \right)$$

where  $\left( \frac{\cdot}{p} \right)$  is the quadratic residue symbol and

$$\sigma_{m, l}(2^a) = \# \{n \pmod{2^{a+2}} : lm^3 - l^2 d^2 n^2 \equiv 2^{a+1} \pmod{2^{a+2}}\} = \begin{cases} 4 & a = 0, l \equiv 2 \pmod{4} \\ 2 & a = 0, lm \equiv 3 \pmod{4} \\ 4 & a \geq 1, lm \equiv 2^{a+1} + 1 \pmod{8} \\ 0 & \text{otherwise} \end{cases}.$$

*Proof.* The lemma follows by the Chinese Remainder Theorem together with the observation that for odd  $p, \alpha \geq 1$ ,

$$\# \{n \pmod{p^\alpha} : lm^3 \equiv l^2 d^2 n^2 \pmod{p^\alpha}\} = 1 + \left( \frac{lm}{p} \right).$$

□

<sup>5</sup>Throughout this section and the next radicals are understood to be zero where the radicand is negative.

Bounding  $\rho_{m,l,d}(s^2t^2) \leq 2^{\omega(s^2t^2)+2} \leq 4\tau(s)\tau(t)$ , the error in expression (6) for  $S_1$  is

$$(7) \quad O(Y^{5/2}Z(\log Z)D^{3/4}(\log D)^2).$$

Next we extract the constant mean value of  $\rho_{m,l,d}(s^2t^2)$  (see [14] Lemma 2). By dropping the error and employing the multiplicative property of  $\rho_{m,l,d}(s^2t^2)$ ,  $S_1$  becomes

$$\frac{1}{4} \sum_{\substack{lt \leq Y^{3/2}D^{1/4} \\ (t,l)=1 \\ l \text{ squarefree} \\ 2^a \parallel t^2}} \frac{1}{t^2} \sum_{\substack{m \leq \frac{Y\sqrt{D}}{l} \\ (m,2lt)=1}} \sum_{d|m} \frac{\mu(d)\sigma_{m,l}(2^a)}{d} \sum_{\substack{s \leq Z \\ (s,2lm)=1}} \frac{\mu(s)}{s^2} \left[ \sqrt{\frac{m^3}{l} - \frac{m^2t^2}{Y^2}} - \sqrt{\frac{m^3}{l} - \frac{t^2D}{l^2}} \right] \sum_{q|st \text{ odd}} \left( \frac{lm}{q} \right).$$

For fixed  $l, t$  with  $2^a \parallel t^2$  set

$$S(z, l, t) = \sum_{\substack{m \leq z \\ (m,2lt)=1}} \sum_{d|m} \frac{\mu(d)\sigma_{m,l}(2^a)}{d} \sum_{\substack{s \leq Z \\ (s,2lm)=1}} \frac{\mu(s)}{s^2} \sum_{q|st \text{ odd}} \left( \frac{lm}{q} \right)$$

so that<sup>6</sup>

$$(8) \quad S_1 = \frac{1}{4} \sum_{\substack{lt \leq Y^{3/2}D^{1/4} \\ l \text{ squarefree} \\ (l,t)=1}} \frac{1}{t^2} \int_{\frac{lt^2}{Y^2}}^{\frac{Y\sqrt{D}}{l}} \left[ \sqrt{\frac{z^3}{l} - \frac{z^2t^2}{Y^2}} - \sqrt{\frac{z^3}{l} - \frac{t^2D}{l^2}} \right] d(S(z, l, t)).$$

**Lemma 4.3.**  $S(z, l, t) = C(l, t)z + A(z)$  with

$$C(l, t) = c_2(l, t) \frac{6}{\pi^2} \prod_p \left( 1 - \frac{1}{p(p+1)} \right) \prod_{p|l} \frac{p^2}{p^2+p-1} \prod_{p|t} \frac{p^2-1}{p^2+p-1}; \quad c_2(l, t) = \begin{cases} 4 & l \text{ even}, t \text{ odd} \\ \frac{4}{5} & l \text{ odd}, t \text{ odd} \\ \frac{4}{3} & l \text{ odd}, t \text{ even} \end{cases}$$

and  $A(z) = O(\tau(l)\tau(t)^2\sqrt{t} \log t \log z + zZ^{-1})$ .

*Proof.* Rearranging order of summation and using characters mod 8 to isolate the value of  $\sigma_{*,l}(2^a)$

$$(9) \quad \begin{aligned} S(z, l, t) &= \sum_{\substack{s \leq Z \\ (s,2l)=1}} \frac{\mu(s)}{s^2} \sum_{\substack{q|st \\ \text{odd}}} \sum_{\substack{d \leq z \\ (d,2slt)=1}} \frac{\mu(d)}{d} \sum_{\substack{m \leq \frac{z}{d} \\ (m,2slt)=1}} \sigma_{m,d,l}(2^a) \left( \frac{lmd}{q} \right) \\ &= \sum_{\substack{s \leq Z \\ (s,2l)=1}} \frac{\mu(s)}{s^2} \sum_{\substack{q|st \\ \text{odd}}} \left( \frac{l}{q} \right) \sum_{\substack{b \bmod 8 \\ \text{odd}}} \sigma_{b,l}(2^a) \frac{1}{4} \sum_{\chi \bmod 8} \overline{\chi(b)} \sum_{\substack{d \leq z \\ (d,2slt)=1}} \frac{\mu(d)\chi(d) \left( \frac{d}{q} \right)}{d} \sum_{\substack{m \leq \frac{z}{d} \\ (m,2slt)=1}} \chi(m) \left( \frac{m}{q} \right). \end{aligned}$$

By Möbius inversion the inner-most sum may be written

$$(10) \quad \sum_{r|2slt} \mu(r)\chi(r) \left( \frac{r}{q} \right) \sum_{m \leq \frac{z}{rd}} \chi(m) \left( \frac{m}{q} \right).$$

<sup>6</sup>The lower evaluation is the point where both radicands become negative.

When  $q = 1$  and  $\chi$  is the trivial character, the sum over  $m$  is  $\frac{z}{2rd} + O(1)$  and we obtain a main contribution to  $S(z, l, t)$  of

$$\begin{aligned} & \frac{1}{4} \sum_{\substack{b \bmod 8 \\ \text{odd}}} \sigma_{b,l}(2^a) \sum_{\substack{s < Z \\ (s, 2l)=1}} \frac{\mu(s)}{s^2} \sum_{\substack{d < z \\ (d, 2slt)=1}} \frac{\mu(d)}{d} \sum_{\substack{r|slt \\ \text{odd}}} \mu(r) \left( \frac{z}{2rd} + O(1) \right) \\ &= \frac{z}{8} \sum_{\substack{b \bmod 8 \\ \text{odd}}} \sigma_{b,l}(2^a) \sum_{\substack{s < Z \\ (s, 2l)=1}} \frac{\mu(s)}{s^2} \sum_{\substack{d \leq z \\ (d, 2silt)=1}} \frac{\mu(d)}{d^2} \sum_{\substack{r|silt \\ \text{odd}}} \frac{\mu(r)}{r} + O(\tau(lt) \log z) \\ &= \frac{z}{8} \sum_{\substack{b \bmod 8 \\ \text{odd}}} \sigma_{b,l}(2^a) \left( O(Z^{-1} + z^{-1}) + \sum_{(s, 2l)=1} \frac{\mu(s)}{s^2} \sum_{(d, 2silt)=1} \frac{\mu(d)}{d^2} \sum_{\substack{r|silt \\ r \text{ odd}}} \frac{\mu(r)}{r} \right) + O(\tau(lt) \log z). \end{aligned}$$

Written as a product, the main sum here matches the main term in the lemma.

When either  $q \neq 1$  or  $\chi$  is not the principal character,  $\chi\left(\frac{\cdot}{q}\right)$  is a non-trivial character of conductor at most  $8q$ . Thus the sum over  $m$  in (10) is bounded by a constant times  $\sqrt{q} \log q$ . Inserting this bound into (9) we find that the contribution from  $\chi \neq \chi_0$  or  $q \neq 1$  is

$$\ll \sum_{s < Z} \frac{1}{s^2} \sum_{q|st} \sqrt{q} \log q \sum_{d \leq z} \frac{1}{d} \sum_{r|slt} 1 = O\left(\tau(l)\tau(t)^2 \sqrt{t} \log t \log z\right).$$

□

We now evaluate the integral in expression (8).

**Lemma 4.4.** Write  $f(z) = \sqrt{\frac{z^3}{l} - \frac{z^2 t^2}{Y^2}} - \sqrt{\frac{z^3}{l} - \frac{t^2 D}{l^2}}$ .

$$(11) \quad S_1 = \frac{1}{4} \sum_{\substack{lt \leq Y^{3/2} D^{1/4} \\ l \text{ squarefree} \\ (l, t)=1}} \frac{C(l, t)}{t^2} \int_{\frac{lt^2}{Y^2}}^{\frac{Y\sqrt{D}}{l}} f(z) dz + O\left(Y^{3/2} D^{3/4} (\log D)^4 + Y D Z^{-1}\right).$$

*Proof.* By Lemma 4.3 we may write  $d(S(z, l, t)) = C(l, t) dz + d(A(z))$ . Integration against the first measure gives the main term of the lemma, so it remains to show that the size of

$$(12) \quad \sum_{\substack{lt \leq Y^{3/2} D^{1/4} \\ l \text{ squarefree} \\ (l, t)=1}} \frac{1}{t^2} \int_{\frac{lt^2}{Y^2}}^{\frac{Y\sqrt{D}}{l}} f(z) d(A(z))$$

is bounded by the error term. It's straightforward to check that the function  $f(z)$  vanishes at  $\frac{lt^2}{Y^2}$  and  $\frac{Y\sqrt{D}}{l}$ , is increasing on  $[\frac{lt^2}{Y^2}, (\frac{t^2 D}{l})^{1/3}]$  and decreasing on  $[(\frac{t^2 D}{l})^{1/3}, \frac{Y\sqrt{D}}{l}]$ , and  $f((\frac{t^2 D}{l})^{1/3}) \leq \frac{t D^{1/2}}{l}$ . Hence, after integration by parts, we obtain

$$\begin{aligned} \left| \int_{\frac{lt^2}{Y^2}}^{\frac{Y\sqrt{D}}{l}} f(z) d(A(z)) \right| &\leq \sup_{z \in [\frac{lt^2}{Y^2}, \frac{Y\sqrt{D}}{l}]} |A(z)| \left( \int_{\frac{lt^2}{Y^2}}^{(\frac{t^2 D}{l})^{1/3}} d(f(z)) - \int_{(\frac{t^2 D}{l})^{1/3}}^{\frac{Y\sqrt{D}}{l}} d(f(z)) \right) \\ &= 2f\left(\left(\frac{t^2 D}{l}\right)^{1/3}\right) \sup_{z \in [\frac{lt^2}{Y^2}, \frac{Y\sqrt{D}}{l}]} |A(z)| \\ &\ll \frac{t D^{1/2}}{l} \left[ \tau(l)\tau(t)^2 \sqrt{t} \log t \log D + \frac{Y\sqrt{D}}{lZ} \right]. \end{aligned}$$

Inserted for the integral in (12), this gives the bound.

□

We now explicitly evaluate the integral in (11) to obtain the main term.

**Lemma 4.5.** *We have*

$$\int_{\frac{1+t^2}{Y^2}}^{\frac{Y\sqrt{D}}{t}} f(z) dz = \frac{1}{3} \frac{D^{\frac{5}{6}} t^{\frac{5}{3}}}{l^{\frac{4}{3}}} \left[ \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{(-1)^n}{\frac{5}{6} - n} \right] - \frac{l^2 t^5}{Y^5} \left[ \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{(-1)^n}{\frac{5}{2} - n} \right] + \sum_{n=1}^{\infty} \left[ \binom{\frac{1}{2}}{n} \frac{(-1)^n l^{2n-3} t^{2n}}{Y^{3n-\frac{5}{2}} D^{\frac{n}{2}-\frac{5}{4}}} \left[ \frac{1}{\frac{5}{2} - n} - \frac{1}{\frac{5}{2} - 3n} \right] \right].$$

*Proof.* We can write the integral as

$$\int_{\frac{1+t^2}{Y^2}}^{\frac{Y\sqrt{D}}{t}} \sqrt{\frac{z^3}{l} - \frac{z^2 t^2}{Y^2}} dz - \int_{(\frac{1+t^2}{l})^{\frac{1}{3}}}^{\frac{Y\sqrt{D}}{t}} \sqrt{\frac{z^3}{l} - \frac{t^2 D}{l^2}} dz = \int_1^{\frac{Y^3 \sqrt{D}}{l^2 t^2}} \left( \frac{l^2 t^5 w}{Y^5} - \frac{1}{3} \frac{D^{\frac{5}{6}} t^{\frac{5}{3}}}{l^{\frac{4}{3}} w^{\frac{2}{3}}} \right) \sqrt{w-1} dw$$

by putting  $w = \frac{Y^2 z}{l t^2}$  in the first term and  $w = \frac{l z^3}{t^2 D}$  in the second. Now expand

$$(w-1)^{1/2} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-1)^n w^{\frac{1}{2}-n}$$

to obtain

$$\int_1^{\frac{Y^3 \sqrt{D}}{l^2 t^2}} \frac{l^2 t^5}{Y^5} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-1)^n w^{\frac{3}{2}-n} - \frac{1}{3} \frac{D^{\frac{5}{6}} t^{\frac{5}{3}}}{l^{\frac{4}{3}}} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-1)^n w^{\frac{1}{6}-n} dw$$

which gives the result after integrating term-by-term. Note that each sum converges absolutely because  $\left| \binom{\frac{1}{2}}{n} \right| \ll n^{-3/2}$ .  $\square$

**Lemma 4.6.** *Let  $F(s) = \sum_{\substack{(l,t)=1 \\ l \text{ squarefree}}} \frac{C(l,t)}{l^{s+1} t^s}$ . For  $r > \frac{-2}{3}$  and  $X > 1$ ,*

$$\sum_{\substack{lt \leq X \\ l \text{ squarefree} \\ (l,t)=1}} C(l,t) l^{r-1} t^r = \frac{48}{\pi^4} \frac{X^{r+1}}{r+1} + \delta_{\{r \leq 0\}} F(-r) + O(X^{r+1/3} \exp(-c(\log X)^{1/3})).$$

Here

$$F(1/3) = \frac{4}{\pi^2} \zeta\left(\frac{1}{3}\right) [1 - 2^{\frac{1}{3}} + 2^{\frac{2}{3}}] \prod_{p \text{ odd}} \left( 1 - \frac{p^{\frac{1}{3}} + 1}{p^2 + p} \right).$$

*Proof.* We have the formulas

$$F(s) = \frac{24}{5\pi^2} \prod_p \left( 1 - \frac{1}{p(p+1)} \right) \left[ 1 + \frac{2}{2^s} + \frac{1}{2^s - 1} \right] \prod_{p \text{ odd}} \left( 1 + \frac{p}{p^2 + p - 1} \frac{1}{p^s} + \frac{p^2 - 1}{p^2 + p - 1} \frac{1}{p^s - 1} \right) \\ G(s) = \frac{F(s)}{\zeta(s)} = \frac{24}{5\pi^2} \prod_p \left( 1 - \frac{1}{p(p+1)} \right) \left[ 1 + \frac{2}{2^s} - \frac{2}{2^{2s}} \right] \prod_{p \text{ odd}} \left( 1 - \frac{p}{p^2 + p - 1} \frac{1}{p^{2s}} \right) \\ \text{Res}_{s=1} F(s) = G(1) = \frac{48}{\pi^4}.$$

Also, the function  $H(s) = \zeta(2s+1)G(s)$  is given by an absolutely convergent Euler product for  $\Re(s) > \frac{-1}{2}$ .

By Perron summation

$$\sum_{\substack{lt \leq X \\ l \text{ squarefree} \\ (l,t)=1}} C(l,t) l^{r-1} t^r = \frac{1}{2\pi i} \int_{(r+1+\frac{1}{\log X})} F(s-r) X^s \frac{ds}{s} = \frac{1}{2\pi i} \int_{(1+\frac{1}{\log X})} \frac{H(s)\zeta(s)}{\zeta(2s+1)} X^{s+r} \frac{ds}{s+r}.$$

In the latter expression, truncate the integral at height  $T$ , shift the line of integration to  $\sigma = -c(\log T)^{-\frac{2}{3}}$ , and delete the part of the resulting contour with imaginary part less than 1. Both the truncated part of the integral and the horizontal segments are bounded by  $O(T^{-1}X^{r+1} \log X)$  (see, for example, [15], II.2) while the deleted segment had size  $O(X^r)$ . In shifting the contour, we pass a pole from  $\zeta$  at  $s = 1$  with residue  $G(1)\frac{X^{r+1}}{r+1}$  and, if  $r \leq c(\log T)^{-\frac{2}{3}}$ , a second pole at  $s = -r$  with residue  $F(-r)$ . Thus

$$\begin{aligned} \sum_{\substack{lt \leq X \\ l \text{ squarefree} \\ (l,t)=1}} C(l,t) l^{r-1} t^r &= \frac{48}{\pi^4} \frac{X^{r+1}}{r+1} + \delta_{\{r \leq 0\}} F(-r) + O(X^r + T^{-1}X^{r+1} \log X) \\ &\quad + X^{r-c(\log T)^{-\frac{2}{3}}} \frac{1}{2\pi} \int_{1 \leq |t| \leq T} \frac{H(\sigma+it)\zeta(\sigma+it)}{\zeta(2\sigma+2-1)} X^{it} \frac{dt}{r+\sigma+it} \end{aligned}$$

On the segment of integration  $H$  is uniformly bounded, while

$$|\zeta(\sigma+it)| \ll \zeta(1-\sigma) \left| \frac{\Gamma(\frac{1-\sigma-it}{2})}{\Gamma(\frac{\sigma+it}{2})} \right| \ll t^{\frac{1}{2}+c(\log t)^{-\frac{2}{3}}} (\log t)^{\frac{2}{3}}, \quad |\zeta(1+2\sigma+2it)|^{-1} \ll \log T$$

so that the integral term is bounded by

$$\ll X^{r-c(\log T)^{-2/3}} (\log T)^{\frac{5}{3}} \int_1^T \exp\left(\frac{-1}{2} \log t + c(\log t)^{\frac{1}{3}}\right) dt \ll X^r T^{\frac{1}{2}} (\log T)^3 \left(\frac{T}{X}\right)^{c(\log T)^{-\frac{2}{3}}}.$$

Choosing  $T = X^{2/3} \exp(c'(\log X)^{1/3})$  gives the result.  $\square$

Dropping the error term in (11) and applying Lemmas 4.5 and 4.6 we obtain

$$\begin{aligned} S_1 &= \frac{1}{4} \left\{ \frac{1}{3} D^{\frac{5}{6}} \left[ \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{(-1)^n}{\frac{5}{6}-n} \right] \sum_{\substack{lt \leq Y^{3/2} D^{1/4} \\ l \text{ squarefree} \\ (l,t)=1}} \frac{C(l,t)}{l^{\frac{4}{3}} t^{\frac{1}{3}}} - \frac{1}{Y^5} \left[ \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{(-1)^n}{\frac{5}{2}-n} \right] \sum_{\substack{lt \leq Y^{3/2} D^{1/4} \\ l \text{ squarefree} \\ (l,t)=1}} C(l,t) l^2 t^3 \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \left[ \binom{\frac{1}{2}}{n} \frac{(-1)^n}{Y^{3n-\frac{5}{2}} D^{\frac{n}{2}-\frac{5}{4}}} \left[ \frac{1}{\frac{5}{2}-n} - \frac{1}{\frac{5}{2}-3n} \right] \sum_{\substack{lt \leq Y^{3/2} D^{1/4} \\ l \text{ squarefree} \\ (l,t)=1}} C(l,t) l^{2n-3} t^{2n-2} \right] \right\} \\ &= \frac{1}{4} \left\{ \left( \frac{24}{\pi^4} YD + \frac{1}{3} F\left(\frac{1}{3}\right) D^{\frac{5}{6}} \right) \left[ \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{(-1)^n}{\frac{5}{6}-n} \right] - \frac{12}{\pi^4} YD \left[ \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{(-1)^n}{\frac{5}{2}-n} \right] \right. \\ (13) \quad &\quad \left. + \frac{48}{\pi^4} YD \left[ \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} \frac{(-1)^n}{2n-1} \left( \frac{1}{\frac{5}{2}-n} - \frac{1}{\frac{5}{2}-3n} \right) \right] \right\} + O\left(D^{\frac{5}{6}} \exp(-c(\log Y^{\frac{3}{2}} D^{\frac{1}{4}})^{\frac{1}{3}})\right) \end{aligned}$$

**Lemma 4.7.** For  $s \in \mathbb{C}$  not equal to  $\frac{3}{2}, \frac{1}{2}$  or half a negative odd integer,

$$I(s) = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{(-1)^n}{\frac{3}{2}-n-s} = \frac{\sqrt{\pi}}{2} \frac{\Gamma(s-\frac{1}{2})}{(\frac{3}{2}-s)\Gamma(s)}.$$

For  $w$  with the same restrictions, and also  $w \neq s$ ,

$$J(w,s) = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{(-1)^n}{(\frac{3}{2}-n-s)(\frac{3}{2}-n-w)} = \frac{\sqrt{\pi}}{2(w-s)} \left[ \frac{\Gamma(w-\frac{1}{2})}{(\frac{3}{2}-w)\Gamma(w)} - \frac{\Gamma(s-\frac{1}{2})}{(\frac{3}{2}-s)\Gamma(s)} \right] = \frac{I(w)-I(s)}{w-s}.$$

*Proof.* For  $\Re(s) > \frac{3}{2}$

$$\begin{aligned} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{(-1)^n}{\frac{3}{2} - n - s} &= - \int_1^{\infty} z^{-s} \sqrt{z-1} dz \\ &= -2 \int_0^{\infty} (u^2 + 1)^{-s} u^2 du = - \int_{-\infty}^{\infty} (u^2 + 1)^{-s+1} du + \int_{-\infty}^{\infty} (u^2 + 1)^{-s} du \\ &= -\sqrt{\pi} \left[ \frac{\Gamma(s - \frac{3}{2})}{\Gamma(s-1)} - \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \right] = \frac{\sqrt{\pi}}{2} \frac{\Gamma(s - \frac{1}{2})}{(\frac{3}{2} - s)\Gamma(s)}. \end{aligned}$$

The first identity then follows for all  $s$  since the two sides define meromorphic functions which agree on a half plane. The second identity is proved in the same way.  $\square$

From Lemma 4.7 it follows  $\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{(-1)^n}{\frac{5}{2} - n} = I\left(\frac{2}{3}\right)$ ,  $\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{(-1)^n}{\frac{3}{2} - n} = I(-1)$  and

$$\begin{aligned} \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} \frac{(-1)^n}{2n-1} \left( \frac{1}{\frac{5}{2} - n} - \frac{1}{\frac{5}{2} - 3n} \right) &= \frac{1}{6} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{(-1)^n}{(\frac{1}{2} - n)(\frac{5}{6} - n)} - \frac{1}{2} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{(-1)^n}{(\frac{1}{2} - n)(\frac{5}{2} - n)} \\ &= \frac{1}{6} J\left(1, \frac{2}{3}\right) - \frac{1}{2} J(1, -1) = \frac{1}{4} I(1) + \frac{1}{4} I(-1) - \frac{1}{2} I\left(\frac{2}{3}\right). \end{aligned}$$

Hence we conclude that the main term of expression (13) is

$$\mathcal{S}_1 = \frac{3I(1)}{4} YD + \frac{F(\frac{1}{3})I(\frac{2}{3})}{12} D^{\frac{5}{6}} = \frac{3}{\pi^3} YD + \frac{\zeta(\frac{1}{3})\Gamma(\frac{1}{6})}{5\pi^{\frac{3}{2}}\Gamma(\frac{2}{3})} \left[ 1 - 2^{\frac{1}{3}} + 2^{\frac{2}{3}} \right] \prod_{p \text{ odd}} \left( 1 - \frac{p^{\frac{1}{3}} + 1}{p^2 + p} \right) D^{\frac{5}{6}}$$

This calculation, combined with the error estimates (7), (11), and (13), proves Proposition 4.1.

**4.2. The upper bound, proof of Proposition 4.3.** This is quite similar to the previous section, so we only provide a sketch.

If  $Y > \frac{2}{\sqrt{3}}$  we may apply Corollary 2.1 to obtain

$$\begin{aligned} S(D, Y) &= \sum_{d \leq D}^* \sum_{\substack{a \in H_3(-d)^* \\ \text{primitive} \\ Na < Y\sqrt{D}}} 1 \ll Y \sum_{d \leq D}^* |H_3(-d)^*| = Y \sum_{d \leq D}^* \sum_{\substack{a \in H_3(-d)^* \\ \text{primitive}}} \mathbf{1}_{\mathcal{F}}(z_a) \\ &\leq \sum_{d \leq D}^* \sum_{\substack{a \in H_3(-d)^* \\ \text{primitive} \\ \Im(z_a) \geq \frac{\sqrt{3}}{2}}} 1 = YS\left(D, \frac{2}{\sqrt{3}}\right) \end{aligned}$$

since  $\mathcal{F} \subset \left\{ z : \Im(z) \geq \frac{\sqrt{3}}{2} \right\}$ , so we may assume  $Y \leq \frac{2}{\sqrt{3}}$  and work from the bound  $S(D, Y) \leq 2\mathcal{S}_3$  (eqn. (4)).

In analogy with equation (5), but dropping the restrictions  $(m, 2n) = 1$  and  $l$  squarefree we obtain

$$\begin{aligned} \mathcal{S}_3 &\leq \sum_{\substack{lt \leq Y^{3/2} D^{1/4} \\ (l, t) = 1}} \sum_{\substack{m \leq \frac{Y\sqrt{D}}{t} \\ (m, lt) = 1}} \sum_{\substack{\frac{m^3}{t} - \frac{t^2 D}{t^2} \leq n^2 \leq \frac{m^3}{t} - \frac{t^2 m^2}{Y^2} \\ lm^3 - l^2 n^2 \equiv 2t^2 \pmod{4t^2}}} 1 \\ &= \sum_{\substack{lt \leq Y^{3/2} D^{1/4} \\ (l, t) = 1}} \sum_{\substack{m \leq \frac{Y\sqrt{D}}{t} \\ (m, lt) = 1}} \rho_{m, l, 1}(t^2) \left[ \frac{1}{t^2} \left( \sqrt{\frac{m^3}{l} - \frac{t^2 m^2}{Y^2}} - \sqrt{\frac{m^3}{l} - \frac{t^2 D}{l^2}} \right) + O(1) \right]. \end{aligned}$$

If  $Y < D^{\frac{1}{6}}$  the sum is empty. Otherwise, using  $\rho_{m,l,d}(t^2) \ll \tau(t) \ll D^\epsilon$  and writing  $lt = q$ ,  $lm = r$  in  $O(\tau(q)\tau(r)) = O(D^\epsilon)$  ways, we have

$$(14) \quad S_3 \ll Y^{\frac{5}{2}} D^{\frac{3}{4}+\epsilon} + D^\epsilon \sum_{q \leq Y^{\frac{3}{2}} D^{\frac{1}{4}}} \sum_{r \leq Y\sqrt{D}} \frac{1}{q^2} \left( \sqrt{r^3 - \frac{r^2 q^2}{Y^2}} - \sqrt{r^3 - q^2 D} \right).$$

Using the bounds

$$\sqrt{r^3 - \frac{r^2 q^2}{Y^2}} - \sqrt{r^3 - q^2 D} \ll \begin{cases} r^{\frac{3}{2}} & 1 \leq r \leq 2q^{\frac{2}{3}} D^{\frac{1}{3}} \\ \frac{q^2 D}{r^{\frac{3}{2}}} & 2q^{\frac{2}{3}} D^{\frac{1}{3}} \leq r \leq Y\sqrt{D} \end{cases}$$

we obtain

$$\ll D^\epsilon \sum_{q \leq Y^{\frac{3}{2}} D^{\frac{1}{4}}} \left\{ \sum_{r \leq 2q^{\frac{2}{3}} D^{\frac{1}{3}}} \frac{r^{\frac{3}{2}}}{q^2} + \sum_{2q^{\frac{2}{3}} D^{\frac{1}{3}} < r \leq Y\sqrt{D}} \frac{D}{r^{\frac{3}{2}}} \right\} \ll Y D^{1+\epsilon}$$

for the sum in (14), which completes the proof of Proposition 4.3.

**4.3. The error term  $S_2$ .** Recall that we defined

$$S_2 = \sum_{\substack{d \leq D \\ d \equiv 2 \pmod{4}}} \sum_{\substack{s^2 | d \\ s \geq Z}} \mu(s) |S(d, Y)|;$$

$$S(d, Y) = \left\{ (l, m, n, t) \in (\mathbb{Z}^+)^4 : lm^3 = l^2 n^2 + t^2 d, l \text{ squarefree}, l | d, (m, ntd) = 1, lm \leq Y\sqrt{d} \right\}.$$

Hence for any  $\epsilon > 0$ ,

$$|S_2| \leq \sum_{\substack{d \leq D \\ d \equiv 2 \pmod{4} \\ q^2 | d \text{ some } q > Z}} \tau(d) |S(d, Y)| \ll D^\epsilon \sum_{\substack{d \leq D \\ d \equiv 2 \pmod{4} \\ q^2 | d \text{ some } q > Z}} |S(d, Y)|$$

so Proposition 4.2 is proved by the following lemma.

**Lemma 4.8.** *We have*

$$\sum_{\substack{d \leq D \\ d \equiv 2 \pmod{4} \\ q^2 | d \text{ some } q > Z}} |S(d, Y)| \ll \frac{Y D^{1+\epsilon}}{Z}.$$

*Proof.* Suppose that  $d$  is counted in the above sum and let  $q > Z$  be maximal with  $q^2 | d$ . If  $(l, m, n, t) \in S(d, Y)$  then  $(ln, tq) = 1$ , since if  $p | (ln, tq)$  then  $p^2 | l^2 n^2 + t^2 d$  so that  $p^2 | lm^3$  which forces  $p | m$ , contradicting  $(m, ntd) = 1$ . Hence we may write  $d = \alpha l q^2$  with  $\alpha l$  squarefree,  $\alpha l \equiv 2 \pmod{4}$ . Since  $lm^3 = l^2 n^2 + q^2 t^2 \alpha l$ , in the ring of integers  $\mathbb{Z}[\sqrt{-\alpha l}]$  we have the factorization of ideals

$$(l)(m^3) = (ln + tq\sqrt{-\alpha l})(ln - tq\sqrt{-\alpha l}).$$

In this ring,  $l$  is a product of ramified primes, so  $(l) = l^2$  and  $(m)^3 = b\bar{b}$  with  $b = l^{-1}(ln + tq\sqrt{-\alpha l})$ ,  $\bar{b} = l^{-1}(ln - tq\sqrt{-\alpha l})$ . Now  $(b, \bar{b})$  divides both  $(m)^3$  and  $(2ln)$ , but  $((m)^3, (2ln)) = (1)$ , so  $(b, \bar{b}) = (1)$ . Hence there exists  $c$  with  $c^3 = b$ . Set

$$A(l, m, n, t) := a = lc.$$

We claim  $a$  is primitive. Indeed,  $b$  is primitive because  $(ln, tq) = 1$ . Also  $b | (m)^3$  so  $b$  is co-prime to the different  $\mathfrak{d}$ . It follows that  $c$  is primitive,  $(c, \mathfrak{d}) = (1)$  and hence  $a$  is primitive. Moreover,  $a^3 = (l)b = (l)(ln + tq\sqrt{-\alpha l})$  is principal. Furthermore,  $a \neq (1)$  or else  $b = l = 1$  would imply  $(ln + tq\sqrt{-\alpha l}) = (1)$ , which is not true. Hence  $a$  is a non-trivial primitive ideal in  $H_3(-\alpha l)$ . Note that  $Na = lm$ .

Now fix  $\alpha$  and  $l$  and  $(1) \neq \mathfrak{a} \in H_3(-\alpha l)$ ,  $\mathfrak{a}$  primitive. Knowing  $\mathfrak{a}$  we can recover  $n$  and  $tq$ , so the map

$$A : \bigcup_{\substack{d \leq D \\ d \equiv 2 \pmod{4} \\ q^2 | d \text{ some } q > Z}} S(d, Y) \rightarrow \bigcup_{\substack{\alpha l \leq \frac{D}{Z^2} \\ \alpha l \equiv 2 \pmod{4} \\ \text{squarefree}}} \left\{ (1) \neq \mathfrak{a} \in H_3(-d) \text{ primitive, } N\mathfrak{a} \leq Y\sqrt{D} \right\}$$

is at most  $O(D^\epsilon)$ -to-1. It follows that

$$(15) \quad \sum_{\substack{d \leq D \\ d \equiv 2 \pmod{4} \\ q^2 | d \text{ some } q > Z}} |S(d, Y)| \ll D^{2\epsilon} \sum_{\substack{r \leq \frac{D}{Z^2} \\ r \equiv 2 \pmod{4} \\ \text{squarefree}}} \# \left\{ (1) \neq \mathfrak{a} \in H_3(-r), \text{ primitive, } N\mathfrak{a} \leq \frac{Y\sqrt{D}}{\sqrt{r}} \sqrt{r} \right\}.$$

Where  $\frac{Y\sqrt{D}}{\sqrt{r}} \leq 1$  we use Corollary 2.1 to deduce

$$\# \left\{ (1) \neq \mathfrak{a} \in H_3(-r), \text{ primitive, } N\mathfrak{a} \leq \frac{Y\sqrt{D}}{\sqrt{r}} \sqrt{r} \right\} \ll \frac{Y\sqrt{D}}{\sqrt{r}} |H_3(-r)|$$

while we split the interval  $Y^2 D < r \leq \frac{D}{Z^2}$  dyadically (if non-empty). Hence (15) is

$$\ll YD^{\frac{1}{2}+2\epsilon} \sum_{r \leq \min(\frac{D}{Z^2}, Y^2 D)} \frac{|H_3(-r)|}{\sqrt{r}} + D^{2\epsilon} \sum_{\frac{Y^2 D}{2} < R = 2^e < \frac{D}{Z^2}} \sum_{R < r \leq 2R} \# \left\{ \mathfrak{a} \in H_3(-r)^*, \text{ primitive, } N\mathfrak{a} \leq \frac{Y\sqrt{D}}{\sqrt{R}} \sqrt{r} \right\}$$

The first sum is  $\ll \frac{YD^{1+2\epsilon}}{Z}$  by the Davenport-Heilbronn theorem and partial summation. By Proposition 4.3, the second is

$$\ll D^{2\epsilon} \sum_{\frac{Y^2 D}{2} < R = 2^e < \frac{D}{Z^2}} S(2R, \frac{Y\sqrt{D}}{\sqrt{R}}) \ll \frac{YD^{1+3\epsilon}}{Z}.$$

□

## 5. HORIZONTAL DISTRIBUTION OF HEEGNER POINTS ATTACHED TO THE 3-PART OF THE CLASS GROUP

In this section we complete the proof that the Heegner points associated to  $H_3(-d)^*$ ,  $0 > -d > -D$  become equidistributed with respect to hyperbolic measure by proving Theorem 1.3.

The initial manipulation of the sum  $S(D, Y; f)$  follows the same steps as for  $S(D, Y)$  in Section 4. In order to apply Proposition 3.1 we first add in the principal ideals other than  $(1)$

$$(16) \quad S(D, Y; f) = \sum_{d \leq D}^* \sum_{\substack{\mathfrak{a} \in H_3(-d)^* \\ \text{primitive} \\ N\mathfrak{a} \leq Y\sqrt{d}}} e(f\Re(N\mathfrak{a})) = O(Y^3) + \sum_{d \leq D}^* \sum_{\substack{(1) \neq \mathfrak{a} \in H_3(-d) \\ \text{primitive} \\ N\mathfrak{a} \leq Y\sqrt{d}}} e(f\Re(N\mathfrak{a})).$$

Recall that we defined

$$S(d, Y) = \left\{ (l, m, n, t) \in (\mathbb{Z}^+)^4 : lm^3 = l^2 n^2 + t^2 d, l \text{ squarefree, } l|d, (m, ntd) = 1, lm \leq Y\sqrt{d} \right\}$$

and that through Proposition 3.1 the tuple  $(l, m, n, t) \in S(d, Y)$  corresponds to the ideal pair

$$(\mathfrak{a}, \bar{\mathfrak{a}}) = \left( [lm, nt^{-1} + \sqrt{-d}], [lm, -nt^{-1} + \sqrt{-d}] \right), \quad tt^{-1} \equiv 1 \pmod{m}$$



with Heegner points  $(z_a, z_{\bar{a}}) = (\frac{nt^{-1} + \sqrt{-d}}{lm}, \frac{-nt^{-1} + \sqrt{-d}}{lm})$ . Hence, sieving for squarefree  $d$ ,

$$\begin{aligned} S(D, Y; f) + O(Y^3) &= 2\Re \sum_{d \leq D}^* S(d, Y; f) = 2\Re \sum_{\substack{d \leq D \\ d \equiv 2 \pmod{4}}} S(d, Y; f) \left\{ \sum_{s^2 | d, s \leq Z} + \sum_{s^2 | d, s > Z} \right\} \mu(s) \\ &= 2\Re(\mathcal{S}_1(f) + \mathcal{S}_2(f)); \quad S(d, Y; f) = \sum_{(l, m, n, t) \in S(d, Y)} e\left(\frac{fnt^{-1}}{m}\right) \end{aligned}$$

From Lemma 4.8 we have the bound

$$|\mathcal{S}_2(f)| \leq \sum_{\substack{d \leq D \\ d \equiv 2 \pmod{4} \\ q^2 | d \text{ some } q > Z}} \tau(d) |S(d, Y; f)| \ll D^\epsilon \sum_{\substack{d \leq D \\ d \equiv 2 \pmod{4} \\ q^2 | d \text{ some } q > Z}} |S(d, Y)| \ll \frac{YD^{1+2\epsilon}}{Z}.$$

**Proposition 5.1.** *For  $D^{-1/6} < Y < D^{1/6}$  we have*

$$\Re(\mathcal{S}_1(f)) \ll ZY^{\frac{5}{2}}D^{\frac{3}{4}+\epsilon} ZfY^{\frac{3}{2}}D^{\frac{3}{4}+\epsilon} + f^{\frac{1}{2}}Y^{\frac{5}{4}}D^{\frac{7}{8}+\epsilon} + f^{\frac{1}{4}}Y^{\frac{11}{8}}D^{\frac{15}{16}+\epsilon}.$$

Putting  $Z = Y^{\frac{3}{4}}D^{\frac{1}{8}}$  gives Theorem 1.3.

**5.1. Bounding  $\Re(\mathcal{S}_1(f))$ : Proof of Proposition 5.1.** The analogue of expression (5) of Section 4.1 is

$$(17) \quad \mathcal{S}_1(f) = \sum_{\substack{lt \leq Y^{3/2}D^{1/4} \\ (l, t)=1 \\ l \text{ squarefree}}} \sum_{\substack{m \leq \frac{Y\sqrt{D}}{l} \\ (m, 2lt)=1}} \sum_{\substack{(n, m)=1 \\ \frac{m^3}{l} - \frac{t^2D}{l^2} \leq n^2 \leq \frac{m^3}{l} - \frac{t^2m^2}{Y^2} \\ lm^3 - l^2n^2 \equiv 2t^2 \pmod{4t^2}}} \sum_{\substack{s^2 | \frac{lm^3 - l^2n^2}{Y^2} \\ s < Z}} \mu(s) e\left(\frac{fnt^{-1}}{m}\right).$$

In order to isolate the long sum over  $n$  we perform Möbius inversion and exchange order of summation to express the final two sums as<sup>7</sup>

$$\sum_{d|m} \mu(d) \sum_{\substack{s < Z \\ (s, 2lm)=1}} \mu(s) \sum_{\substack{\frac{1}{d} \sqrt{\frac{m^3}{l} - \frac{t^2D}{l^2}} \leq n \leq \frac{1}{d} \sqrt{\frac{m^3}{l} - \frac{t^2m^2}{Y^2}} \\ lm^3 - l^2n^2d^2 \equiv 2s^2t^2 \pmod{4s^2t^2}}} e\left(\frac{fndt^{-1}}{m}\right).$$

Writing  $n = 4ks^2t^2 + r$ ,  $0 \leq r < 4s^2t^2$  the sum over  $n$  becomes

$$(18) \quad \sum_{\substack{\frac{\sqrt{\frac{m^3}{l} - \frac{t^2D}{l^2}}}{4ds^2t^2} < k < \frac{\sqrt{\frac{m^3}{l} - \frac{t^2m^2}{Y^2}}}{4ds^2t^2}}} e\left(\frac{4fds^2tk}{m}\right) \sum_{\substack{r \pmod{4s^2t^2} \\ lm^3 - l^2r^2d^2 \\ \equiv 2s^2t^2 \pmod{4s^2t^2}}} e\left(\frac{fdrt^{-1}}{m}\right) + O(\rho_{m, l, d}(s^2t^2)).$$

As before, the contribution to  $\mathcal{S}_1(f)$  from the error  $O(\rho_{m, l, d}(s^2t^2))$  is

$$(19) \quad O(ZY^{5/2}D^{3/4+\epsilon})$$

and we drop this error term.

We now split the sum according to the size of  $m$ . When  $m \ll fds^2t$  we get substantial cancellation from the summation over  $k$ ; otherwise we use the fact that the sum over  $k$  varies smoothly with  $m$  and get cancellation from variation in the sum over  $r$ .

**Proposition 5.2.** *The contribution to  $\mathcal{S}_1(f)$  from  $m < 8dfs^2t$  is*

$$O(\tau(f)D^{1/2+\epsilon} + ZY^{\frac{5}{2}}D^{\frac{3}{4}+\epsilon} + ZfY^{\frac{3}{2}}D^{\frac{3}{4}+\epsilon} + f^{\frac{1}{2}}Y^{\frac{5}{4}}D^{\frac{7}{8}+\epsilon} + f^{\frac{1}{4}}Y^{\frac{11}{8}}D^{\frac{15}{16}+\epsilon}).$$

**Proposition 5.3.** *The contribution to  $\mathcal{S}_1(f)$  from  $m > 8fs^2t$  is*

$$O(\tau(f)Y^{\frac{7}{4}}D^{\frac{3}{4}+\epsilon} + \frac{\tau(f)}{f}Y^{\frac{3}{2}}D^{\frac{5}{8}+\epsilon}).$$

<sup>7</sup>We continue to use the convention that radicals are zero when the radicand is negative.

Combined, these propositions prove Proposition 5.1.

5.1.1. *Small m case: Proof of Proposition 5.2.* Throughout we bound the sum over  $r$  in (18) by  $d^\epsilon$ , so that after exchanging order of summation we need to bound

$$(20) \quad D^\epsilon \sum_{\substack{lt \leq Y^{3/2} D^{1/4} \\ (l,t)=1 \\ l \text{ squarefree}}} \sum_{\substack{s < Z \\ (s,2l)=1}} \sum_{\substack{d \leq \frac{Y\sqrt{D}}{l} \\ (d,2lst)=1}} \sum_{\substack{m \leq \min(8fs^2t, \frac{Y\sqrt{D}}{ld}) \\ (m,2lst)=1}} \left| \sum_{\substack{\sqrt{\frac{d^3 m^3 - t^2 D}{l^2}} < k < \sqrt{\frac{d^3 m^3 - t^2 m^2 d^2}{Y^2}}}} e\left(\frac{4fs^2tk}{m}\right) \right|.$$

The range of the summation in  $k$  is bounded elementarily as follows.

**Lemma 5.1.** *We have the bounds*

$$\sqrt{\frac{m^3 d^3}{l} - \frac{t^2 m^2 d^2}{Y^2}} - \sqrt{\frac{m^3 d^3}{l} - \frac{t^2 D}{l^2}} \ll \begin{cases} a. \frac{tmd}{Y} & m < \frac{2t^2 l}{dY^2} \\ b. \frac{t^2 D}{m^{\frac{3}{2}} l^{\frac{3}{2}} d^{\frac{3}{2}}} & m > \frac{2t^2 l}{dY^2} \\ c. \frac{m^{\frac{3}{2}} d^{\frac{3}{2}}}{l^{\frac{1}{2}}} & \text{all } m \\ d. \frac{tD^{\frac{1}{2}}}{l} & \text{all } m \end{cases}.$$

Applying bound d of Lemma 5.1, the contribution to (20) from  $m$  dividing  $f$  is

$$(21) \quad O(\tau(f) D^{1/2+\epsilon}).$$

For  $m$  in the range  $m < \min(\frac{2lt^2}{dY^2}, 8fs^2t)$  use bound a to bound the summation over  $k$  by  $\frac{m}{4Ys^2t} + O(1)$ . The  $O(1)$  error contributes  $O(ZY^{5/2}D^{3/4+\epsilon})$  as in (19). Since  $\min(a, b) \leq \sqrt{ab}$  we can extend the range of summation to  $m \ll \frac{f^{\frac{1}{2}} l^{\frac{1}{2}} t^{\frac{3}{2}} s}{d^{\frac{1}{2}} Y}$  so that the contribution of the rest is bounded by

$$(22) \quad \begin{aligned} &\ll D^\epsilon \sum_{lt \leq Y^{3/2} D^{3/4}} \sum_{s < Z} \sum_{ld \leq Y\sqrt{D}} \sum_{\substack{m \ll \frac{f^{\frac{1}{2}} l^{\frac{1}{2}} t^{\frac{3}{2}} s}{d^{\frac{1}{2}} Y}}} \frac{m}{Ys^2t} \ll D^\epsilon f Y^{-3} \sum_{lt \leq Y^{3/2} D^{1/4}} lt^2 \sum_{s < Z} \sum_{ld \leq Y\sqrt{D}} \frac{1}{d} \\ &\ll f Z Y^{3/2} D^{3/4+\epsilon} \end{aligned}$$

Where  $m < \frac{t^{\frac{2}{3}} D^{\frac{1}{3}}}{l^{\frac{1}{3}} d}$  apply the bound c on the length of the sum over  $k$  to obtain a contribution

$$\ll D^\epsilon \sum_{lt \leq Y^{3/2} D^{1/4}} \sum_{s < Z} \sum_{ld < Y\sqrt{D}} \sum_{\substack{m < \frac{t^{\frac{2}{3}} D^{\frac{1}{3}}}{l^{\frac{1}{3}} d} \\ \min(\frac{t^{\frac{2}{3}} D^{\frac{1}{3}}}{l^{\frac{1}{3}} d}, 8fs^2t)}} \left( \frac{m^{3/2} d^{1/2}}{s^2 t^2 l^{1/2}} + O(1) \right)$$

Again the  $O(1)$  contributes  $O(ZY^{5/2}D^{3/4+\epsilon})$ . In the rest of the sum, we exchange the sum over  $m$  and  $d$  and replace the bound  $m \leq \min(\frac{t^{\frac{2}{3}} D^{\frac{1}{3}}}{l^{\frac{1}{3}} d}, 8fs^2t)$  with  $m \ll \frac{f^{\frac{1}{2}} s t^{\frac{5}{6}} D^{\frac{1}{6}}}{l^{\frac{1}{6}}}$  for a contribution of

$$(23) \quad \ll D^\epsilon \sum_{lt \leq Y^{\frac{3}{2}} D^{\frac{1}{4}}} \frac{1}{t^2 l^{\frac{1}{2}}} \sum_{s < Z} \frac{1}{s^2} \sum_{\substack{m \ll \frac{f^{\frac{1}{2}} s t^{\frac{5}{6}} D^{\frac{1}{6}}}{l^{\frac{1}{6}}} \\ d < \frac{t^{\frac{2}{3}} D^{\frac{1}{3}}}{l^{\frac{1}{3}} m}}} m^{\frac{3}{2}} \sum d^{\frac{1}{2}} \ll f^{\frac{1}{2}} Y^{\frac{5}{4}} D^{\frac{7}{8}+\epsilon}.$$

Finally, for  $\max(\frac{2lt^2}{dY^2}, \frac{t^{\frac{2}{3}}D^{\frac{1}{3}}}{l^{\frac{1}{3}}d}) < m < 8fs^2t$  use the bound  $b$  together with the usual bound for the sum of a geometric series to deduce<sup>8</sup> a contribution

$$\ll D^\epsilon \sum_{lt \leq Y^{3/2}D^{1/4}} \sum_{s < Z} \sum_{d \leq Y\sqrt{D}} \sum_{\substack{t^{2/3}D^{1/3} < m \\ l^{1/3}d < \min(\frac{Y\sqrt{D}}{d}, 8fs^2t) \\ (m, 2lst)=1 \\ m \nmid f}} \min \left( \left\| \frac{4fs^2t}{m} \right\|^{-1}, \frac{D}{d^{5/2}s^2m^{3/2}l^{3/2}} + O(1) \right).$$

Here one can split the sum over  $s$  according as  $s < Z_1$  and  $Z_1 < s < Z$ . For the large  $s$ , use the second term in the minimum to obtain  $\ll \frac{YD^{1+\epsilon}}{Z_1} + ZY^{\frac{5}{2}}D^{\frac{3}{4}+\epsilon}$ . For small  $s$ , execute the sum over  $t$  first using the estimate

$$\sum_{t < T, (t, m)=1} \left\| \frac{\alpha t}{m} \right\|^{-1} \ll (m+T) \log m \quad m \nmid \alpha$$

to obtain

$$\ll D^\epsilon \sum_{l \leq Y^{3/2}D^{1/4}} \sum_{d \leq Y\sqrt{D}} \sum_{s < Z_1} \sum_{m < \min(8fs^2Y^{3/2}D^{1/4}, \frac{Y\sqrt{D}}{d})} m + l^{-1}Y^{3/2}D^{1/4} \ll fZ_1^3Y^{5/2}D^{3/4+\epsilon}.$$

Choosing  $Z_1 = f^{-1/4}Y^{-3/8}D^{1/16}$  gives a combined error of

$$(24) \quad O(f^{1/4}Y^{11/8}D^{15/16+\epsilon}).$$

Combining estimates (21) through (24) proves Proposition 5.2.

**5.1.2. Large  $m$  case: Proof of Proposition 5.3.** In order to execute the sum over  $m$  in (16) we replace the sum over  $k$  in (17) with a smooth function of  $m$  and the sum over  $r$  with a complete sum  $\text{mod } s^2t^2$ .

**Lemma 5.2.** For  $0 < x < \frac{1}{2}$ ,  $\sum_{\alpha < k < \beta} e(kx) = \frac{e(\alpha x) - e(\beta x)}{2\pi i x} + O(1)$ .

*Proof.* By summing the geometric series,  $\sum_{\alpha < k < \beta} e(kx) = \frac{e(\lceil \alpha \rceil x) - e(\lceil \beta \rceil x)}{e(x) - 1}$ . Now

$$\frac{e(\alpha x) - e(\lceil \alpha \rceil x)}{e(x) - 1} = O(1)$$

and similarly for  $\beta$ . Also,  $e(x) - 1 = 2\pi i x + O(x^2)$ , so  $(e(x) - 1)^{-1} = (2\pi i x)^{-1} + O(1)$ , which proves the lemma.  $\square$

**Lemma 5.3.** Write  $t^{-1}$  for the inverse of  $t \bmod m$ ,  $m^{-1}$  for the inverse of  $m \bmod t$ . We have  $e(\frac{\alpha t^{-1}}{m}) = e(\frac{-\alpha m^{-1}}{t}) + O(\frac{\alpha}{mt})$ .

*Proof.* Let  $tt^{-1} = 1 + jm$  and reduce modulo  $t$  to find that  $j \equiv -m^{-1} \bmod t$ . Hence  $e(\frac{\alpha t^{-1}}{m}) = e(\frac{\alpha tt^{-1}}{mt}) = e(\frac{\alpha}{mt})e(\frac{-\alpha m^{-1}}{t})$ . Finally,  $e(\frac{\alpha}{mt}) - 1 = O(\frac{\alpha}{mt})$ .  $\square$

Applying Lemmas 5.2 and 5.3 in (18), the sum over  $k$  is equal to

$$\frac{m}{8\pi i f d s^2 t} \left[ e \left( f \sqrt{\frac{m}{lt^2} - \frac{1}{Y^2}} \right) - e \left( f \sqrt{\frac{m}{lt^2} - \frac{D}{l^2 m^2}} \right) \right] + O(1)$$

while the sum over  $r$  is equal to

$$\sum_{\substack{r \bmod 4s^2t^2 \\ lm^3 - l^2d^2r^2 \\ \equiv 2s^2t^2 \bmod 4s^2t^2}} e\left(\frac{-frdm^{-1}}{t}\right) + O(\rho_{m,l,d}(s^2t^2))$$

<sup>8</sup>Here  $\|x\|$  denotes the distance from  $x$  to the nearest integer.

so that (18) is given by

$$\frac{m}{8\pi i f d s^2 t} \left[ e \left( f \sqrt{\frac{m}{l t^2} - \frac{1}{Y^2}} \right) - e \left( f \sqrt{\frac{m}{l t^2} - \frac{D}{l^2 m^2}} \right) \right] \sum_{\substack{r \bmod 4 s^2 t^2 \\ l m^3 - l^2 d^2 r^2 \\ \equiv 2 s^2 t^2 \bmod 4 s^2 t^2}} e \left( \frac{-f r d m^{-1}}{t} \right) + O(\rho_{m,l,d}(s^2 t^2)).$$

As before, the error contributes  $O(ZY^{\frac{5}{2}}D^{\frac{3}{4}})$  to  $S_1(f)$ . Dropping the error, and passing summations over  $s$  and  $d$  before the sum over  $m$ , we are left to bound

$$\Re \left\{ \frac{1}{8\pi i f} \sum_{\substack{lt \leq Y^{3/2} D^{1/4} \\ (l,t)=1 \\ l \text{ squarefree}}} \frac{1}{t} \sum_{\substack{s \leq Z \\ (s,2l)=1}} \frac{\mu(s)}{s^2} \sum_{\substack{d \leq \frac{Y\sqrt{D}}{l} \\ (d,2lt)=1}} \sum_{\substack{8fs^2t < m \leq \frac{Y\sqrt{D}}{ld} \\ (m,2lt)=1}} m \left[ e \left( f \sqrt{\frac{md}{lt^2} - \frac{1}{Y^2}} \right) - e \left( f \sqrt{\frac{md}{lt^2} - \frac{D}{l^2 m^2 d^2}} \right) \right] \sum_{\substack{r \bmod 4 s^2 t^2 \\ l m^3 d^3 - l^2 r^2 d^2 \\ \equiv 2 s^2 t^2 \bmod 4 s^2 t^2}} e \left( \frac{-f r m^{-1}}{t} \right) \right\}.$$

After a Möbius inversion to eliminate the condition  $(m, l) = 1$ , and setting apart  $f_1 = (f, t)$ , this is

$$(25) \quad \Re \left\{ \frac{1}{8\pi i f} \sum_{f_1 f_2 = f} \sum_{\substack{f_1 l t \leq Y^{\frac{3}{2}} D^{\frac{1}{4}} \\ (l, f_1 t) = 1 \\ l \text{ squarefree} \\ (t, f_2) = 1}} \frac{1}{f_1 t} \sum_{\substack{s \leq Z \\ (s, 2l) = 1}} \frac{\mu(s)}{s^2} \sum_{\substack{d < \frac{Y\sqrt{D}}{l} \\ (d, 2l s f_1 t) = 1}} \mu(d) \sum_{q|l, 2f_1 q} q \mu(q) \sum_{\substack{8fs^2 f_1 t q^{-1} \leq m \leq \frac{Y\sqrt{D}}{ld} \\ (m, 2s f_1 t) = 1}} m \left[ e \left( f \sqrt{\frac{mqd}{l f_1^2 t^2} - \frac{1}{Y^2}} \right) - e \left( f \sqrt{\frac{mqd}{l f_1^2 t^2} - \frac{D}{l^2 m^2 q^2 d^2}} \right) \right] \sum_{\substack{r \bmod 4 s^2 f_1^2 t^2 \\ l d^3 m^3 q^3 - l^2 r^2 d^2 \equiv 2 s^2 f_1^2 t^2 \bmod 4 s^2 f_1^2 t^2}} e \left( \frac{f_2 r m^{-1} q^{-1}}{t} \right) \right\}.$$

Note that we are now only concerned with the imaginary part of the summation over  $m$  and  $r$ .

Define, for positive integers  $t, A, l$ ,  $(l, At) = 1$  and  $\alpha, \beta$  reduced residues mod  $4At^2$

$$S(z; t, A, l) = \sum_{\substack{m < Z \\ (m, 2At) = 1}} \sum_{\substack{r \bmod 4At^2 \\ \alpha l m^3 - l^2 r^2 \equiv 2At^2 \bmod 4At^2}} e \left( \frac{\beta r m^{-1}}{t} \right),$$

so that imaginary part of the summation over  $m$  and  $r$  in (25) is given by

$$(26) \quad \Im \left\{ \int_{\frac{8fs^2 f_1 t}{q}}^{\frac{Y\sqrt{D}}{ld}} z \left[ e \left( f \sqrt{\frac{z q d}{l f_1^2 t^2} - \frac{1}{Y^2}} \right) - e \left( f \sqrt{\frac{z q d}{l f_1^2 t^2} - \frac{D}{l^2 z^2 q^2 d^2}} \right) \right] d(S(z; t, s^2 f_1^2, l)) \right\}$$

with  $\alpha = d^3 q^3$  and  $\beta = -f_2 m^{-1} d^{-1} q^{-1}$ .

**Lemma 5.4.** *Let  $t, l$ , and  $A$  be positive integers with  $(l, At) = 1$  and let  $\alpha, \beta$  be reduced residues mod  $4At^2$ . We have*

$$S(z; t, A, l) = C_2(l, At) \frac{z}{4At^2} \frac{\mu(t)\phi(At^2)}{\phi(t)} + O((At)^{\frac{1}{2}} \tau(At)^2 \log(At)); \quad C_2(l, t, A) = \begin{cases} 8 & l \text{ even} \\ 2 & l, At \text{ odd} \\ 4 & At \text{ even} \end{cases}.$$

*Proof.* By breaking the sum over  $m$  into blocks of length  $4At^2$ ,  $S = \lfloor \frac{z}{4At^2} \rfloor F + E$  with  $F$  the full sum<sup>9</sup>

$$F = \sum_{m \bmod 4At^2}^* \sum_{\substack{r \bmod 4At^2 \\ \alpha lm^3 - l^2 r^2 \\ \equiv 2At^2 \bmod 4At^2}} e\left(\frac{\beta rm^{-1}}{t}\right)$$

and  $E$  the short sum

$$E = \sum_{\substack{m < y \\ (m, 2At) = 1}} \sum_{\substack{r \bmod 4At^2 \\ \alpha lm^3 - l^2 r^2 \\ \equiv 2At^2 \bmod 4At^2}} e\left(\frac{\beta rm^{-1}}{t}\right); \quad y = z - 4At^2 \left\lfloor \frac{z}{4At^2} \right\rfloor.$$

To evaluate  $F$ , note that when  $l$  is even the condition  $\alpha lm^3 - l^2 r^2 \equiv 2 \bmod 4$  in  $F$  is guaranteed so

$$F = 8 \sum_{\substack{m, r \bmod 2At^2 \\ \alpha lm^3 \equiv l^2 r^2 \bmod 2At^2}}^* e\left(\frac{\beta rm^{-1}}{t}\right).$$

When  $l$  is odd, write  $m = m_1 + 2At^2 m_2$ ,  $r = r_1 + 2At^2 r_2$  with  $0 \leq m_1, r_1 < 2At^2$ ,  $(m_1 r_1, 2At) = 1$  and  $m_2, r_2 \in \{0, 1\}$  so that

$$F = \sum_{\substack{0 \leq m_1, r_1 < 2At^2 \\ (m_1 r_1, 2At) = 1 \\ \alpha lm_1^3 \equiv l^2 r_1^2 \bmod 2At^2}} e\left(\frac{\beta r_1 m_1^{-1}}{t}\right) \sum_{\substack{m_2, r_2 \in \{0, 1\} \\ \alpha lm_1^3 + 6\alpha lm_1^2 m_2 At^2 - l^2 r_1^2 \\ \equiv 2At^2 \bmod 4At^2}} 1 = 2 \sum_{\substack{m, r \bmod 2At^2 \\ \alpha lm^3 \equiv l^2 r^2 \bmod 2At^2}}^* e\left(\frac{\beta rm^{-1}}{t}\right)$$

since

$$\alpha lm_1^3 + 6\alpha lm_1^2 m_2 At^2 - l^2 r_1^2 \equiv 2At^2 \bmod 4At^2$$

is independent of  $r_2$ , but has a unique solution for  $m_2 \in \{0, 1\}$ . After making the change of variables  $m' := \alpha lm$ ,  $r' := \alpha l^2 r$ , and  $w = (m')^{-1} r$  so that  $w^2 \equiv m'$ ,  $w^3 \equiv r'$ ,

$$\sum_{\substack{m', r' \bmod 2At^2 \\ (m')^3 \equiv (r')^2 \bmod 2At^2}}^* e\left(\frac{\beta l(m')^{-1}(r')}{t}\right) = \sum_{w \bmod 2At^2}^* e\left(\frac{\beta lw}{t}\right) = \frac{\phi(2At^2)}{\phi(t)} \mu(t).$$

Hence  $F = C_2(l, At) \frac{\phi(At^2)\mu(t)}{\phi(t)}$  and  $\lfloor \frac{z}{4At^2} \rfloor F = (\frac{z}{4At^2} + O(1)) F$  gives the main term in the lemma.

To bound  $E$  we expand the condition  $m < y$  in additive characters. When  $l$  is even write

$$\begin{aligned} E &= \sum_{m \bmod At^2}^* \sum_{m_1=0}^3 \sum_{\substack{r \bmod 4At^2 \\ \alpha lm^3 \equiv l^2 r^2 \bmod At^2}} e\left(\frac{\beta m^{-1} r}{t}\right) \frac{1}{4At^2} \sum_{x \bmod 4At^2} \sum_{0 \leq b < y \text{ odd}} e\left(\frac{x(b - (m + At^2 m_1))}{4At^2}\right) \\ &= \frac{4}{At^2} \sum_{x \bmod At^2} \sum_{0 \leq b < y \text{ odd}} e\left(\frac{bx}{At^2}\right) \sum_{\substack{m, r \bmod At^2 \\ \alpha lm^3 \equiv l^2 r^2 \bmod At^2}}^* e\left(\frac{At\beta m^{-1} r - xm}{At^2}\right). \end{aligned}$$

In the inner sum, make the change of variables  $m' = \alpha lm = w^2$ ,  $r' = \alpha l^2 r = w^3$ , so that the sum becomes

$$\sum_{w \bmod At^2}^* e\left(\frac{(At\beta l)w - (x\alpha^{-1}l^{-1})w^2}{At^2}\right) = \sum_{w_1 \bmod At}^* \sum_{0 \leq w_2 < t} e\left(\frac{(At\beta l)(w_1 + Atw_2) - (x\alpha^{-1}l^{-1})(w_1^2 + 2Atw_1w_2)}{At^2}\right).$$

<sup>9</sup>In this proof only we use  $\sum^*$  to denote a sum over reduced residues.

The inner sum vanishes unless  $t|x$  so we set  $q = \frac{At^2}{(x, At^2)}$  and deduce

$$E = \frac{4}{At} \sum_{q|At} \sum_{x \bmod q}^* \sum_{0 \leq b < y \text{ odd}} e\left(\frac{bx}{q}\right) \sum_{w \bmod At}^* e\left(\frac{\beta lw}{t} - \frac{x\alpha^{-1}l^{-1}w^2}{q}\right)$$

$$|E| \ll \sum_{q|At} \frac{1}{[q, t]} \sum_{x \bmod q}^* \left| \sum_{0 \leq b < y \text{ odd}} e\left(\frac{bx}{q}\right) \right|_{x \bmod q, (x, q)=1} \sup_{x \bmod q, (x, q)=1} \left| \sum_{w \bmod [q, t]}^* e\left(\frac{\beta lw}{t} - \frac{(x\alpha^{-1}l^{-1})w^2}{q}\right) \right|.$$

By the usual bound for a geometric series, this is

$$\ll \log(At) \sum_{q|At} \frac{q}{[q, t]} \sup_{x \bmod q, (x, q)=1} \left| \sum_{w \bmod [q, t]}^* e\left(\frac{\beta lw}{t} - \frac{(x\alpha^{-1}l^{-1})w^2}{q}\right) \right|.$$

In the case  $l$  is odd, write instead

$$E = \frac{1}{4At^2} \sum_{x \bmod 4At^2} \sum_{0 \leq b < y} e\left(\frac{bx}{4At^2}\right) \sum_{\substack{m, r \bmod 4At^2 \\ \alpha lm^3 - l^2 r^2 \\ \equiv 2At^2 \bmod 4At^2}} e\left(\frac{4At\beta m^{-1}r - xm}{4At^2}\right)$$

and make the change of variables  $m' = \alpha lm = w^2 + 2At^2$ ,  $r' = \alpha l^2 r = w^3 + 2At^2$ . Here, the inner sum vanishes unless  $2t|x$ , and we obtain the similar bound

$$|E| \ll \sum_{q|2At} \frac{1}{[q, t]} \sum_{x \bmod q}^* \left| \sum_{0 \leq b < y} e\left(\frac{bx}{q}\right) \right|_{x \bmod q, (x, q)=1} \sup_{x \bmod q, (x, q)=1} \left| \sum_{w \bmod [q, t]}^* e\left(\frac{\beta lw}{t} - \frac{(x\alpha^{-1}l^{-1})w^2}{q}\right) \right|$$

$$\ll \log(At) \sum_{q|2At} \frac{q}{[q, t]} \sup_{x \bmod q, (x, q)=1} \left| \sum_{w \bmod [q, t]}^* e\left(\frac{\beta lw}{t} - \frac{(x\alpha^{-1}l^{-1})w^2}{q}\right) \right|.$$

In either the case  $l$  is even or odd, the proof is completed by bounding the sum in the supremum by  $\sqrt{[q, t]}\tau([q, t])$  (see [11] Chapter 12).  $\square$

Inserting the result of Lemma 5.4 into the integral in (26) we have

$$S(z; t, s^2 f_1^2, l) = \frac{C_2(l, ts^2 f_1^2)z}{4s^2 f_1^2 t^2} \mu(t) \frac{\phi(s^2 f_1^2 t^2)}{\phi(t)} + E(z; t, s^2 f_1^2, l); \quad E(z; t, s^2 f_1^2, l) = O(sf_1 t^{\frac{1}{2}} \tau(s^2 f_1^2 t)^2 \log D)$$

so that the imaginary part of (26) is bounded by a constant times

$$(27) \quad \int_{\left(\frac{lt^2 f_1^2}{qdY^2}, \frac{\sqrt{YD}}{qld}\right)}^{\max} z \left[ e\left(f \sqrt{\frac{zqd}{lf_1^2 t^2} - \frac{1}{Y^2}}\right) - e\left(f \sqrt{\frac{zqd}{lf_1^2 t^2} - \frac{D}{l^2 q^2 d^2 z^2}}\right) \right] \left( \frac{dz}{t} + d(E(z; t, s^2 f_1^2, l)) \right)$$

Let  $g_1(w)$  and  $g_2(w)$  be the functions such that

$$g_1^{-1}(z) = \sqrt{\frac{zqd}{lf_1^2 t^2} - \frac{1}{Y^2}}, \quad g_2^{-1}(z) = \sqrt{\frac{zqd}{lf_1^2 t^2} - \frac{D}{l^2 q^2 d^2 z^2}}.$$

We collect in the following lemma the facts that we need regarding  $g_1$  and  $g_2$ .

**Lemma 5.5.** Assume  $f_1 lt \leq Y^{3/2} D^{1/4}$  and  $Y < D^{1/6}$ . The functions  $g_1$  and  $g_2$  satisfy

(1)  $g_i, g'_i$ , and  $g''_i$  are positive functions on  $\mathbb{R}^+$

(2)  $g'_2(w) \leq g'_1(w) = 2 \frac{lf_1^2 t^2}{qd} w$

$$(3) \ g_1\left(\frac{1}{D}\right) \ll \frac{l f_1^2 t^2}{q d Y^2}$$

$$(4) \ g_2\left(\frac{1}{D}\right) \ll \frac{D^{\frac{1}{3}} f_1^{\frac{2}{3}} t^{\frac{2}{3}}}{l^{\frac{1}{3}} q d}$$

$$(5) \ \sup_{0 < w} \frac{g_2'(w)}{g_2^2(w)} \ll \frac{q d l}{D^{1/2}}$$

$$(6) \ g_i^{-1}\left(\frac{Y \sqrt{D}}{l q d}\right) \leq \frac{Y^{1/2} D^{1/4}}{l f_1 t}$$

*Proof.* Note  $g_1(w) = \frac{l f_1^2 t^2}{q d} (w^2 - Y^{-2})$ . Items (1) and (2) may be verified by calculus. Items (3) and (4) follow by substituting  $\frac{1}{D}$  in the definitions of  $g_i^{-1}(z)$  and solving for  $z$ . For (5), bound  $g_2'(w)$  by  $\frac{2 l f_1^2 t^2}{q d} w$  and  $w$  by  $\sqrt{\frac{z q d}{l f_1^2 t^2}}$ . The minimum of  $z^{-3/2}$  occurs at the minimum possible value,  $z = g_2(0) = \frac{D^{1/3} f_1^{2/3} t^{2/3}}{l^{1/3} q d}$ . (6) is immediate from the definition.  $\square$

**Proposition 5.4.** *Let  $D^{-\frac{1}{6}} < Y < D^{\frac{1}{6}}$  and  $l f_1 t < Y^{\frac{3}{2}} D^{\frac{1}{4}}$ . The imaginary part of the summation over  $m$  and  $r$  in (25) is bounded by*

$$\text{expr.}(27) \ll \frac{Y^{\frac{3}{2}} D^{\frac{3}{4}} f_1^2}{f^2 l q^2 d^2} + \frac{Y D^{\frac{1}{2} + \epsilon} f_1 s t^{\frac{1}{2}}}{l q d} + \frac{f Y^{\frac{3}{2}} D^{\frac{3}{4} + \epsilon} s}{l^2 q d t^{\frac{1}{2}}}.$$

*Proof.* We first treat the integration with respect to  $\frac{dz}{t}$ . By linearity, separating the two exponentials, the integral is the difference of two integrals. It suffices to consider only the parts of the integrals where the radicands are non-negative, since the remainder contributes a real quantity. In the first integral substitute  $z = g_1(w)$  and in the second  $z = g_2(w)$  to obtain

$$\begin{aligned} & \Im \int_{\left(\frac{l t^2 f_1^2}{q d Y^2}, \max\left(\frac{Y \sqrt{D}}{l q d}, 8 f d s^2 f_1 t\right)\right)}^{\frac{Y \sqrt{D}}{q d}} z \left[ e \left( f \sqrt{\frac{z q d}{l f_1^2 t^2} - \frac{1}{Y^2}} \right) - e \left( f \sqrt{\frac{z q d}{l f_1^2 t^2} - \frac{D}{l^2 q^2 d^2 z^2}} \right) \right] \frac{dz}{t} \\ &= \Im \left\{ \frac{1}{t} \int_{g_1^{-1}\left(\max\left(\frac{l t^2 f_1^2}{q d Y^2}, 8 f d s^2 f_1 t\right)\right)}^{g_1^{-1}\left(\frac{Y \sqrt{D}}{l q d}\right)} g_1(w) g_1'(w) e(f w) dw - \frac{1}{t} \int_{g_2^{-1}\left(\max\left(\frac{D^{1/3} f_1^{2/3} t^{2/3}}{l^{1/3} q d}, 8 f d s^2 f_1 t\right)\right)}^{g_2^{-1}\left(\frac{Y \sqrt{D}}{l q d}\right)} g_2(w) g_2'(w) e(f w) dw \right\}. \end{aligned}$$

Integrate by parts to deduce that the above is

$$\begin{aligned} & \Im \left\{ \frac{1}{f t} g_1(w) g_1'(w) e(f w) \Big|_{g_1^{-1}\left(\max\left(\frac{l t^2 f_1^2}{q d Y^2}, 8 f d s^2 f_1 t\right)\right)}^{g_1^{-1}\left(\frac{Y \sqrt{D}}{l q d}\right)} - \frac{1}{f t} g_2(w) g_2'(w) e(f w) \Big|_{g_2^{-1}\left(\max\left(8 f d s^2 f_1 t, \frac{D^{1/3} f_1^{2/3} t^{2/3}}{l^{1/3} q d}\right)\right)}^{g_2^{-1}\left(\frac{Y \sqrt{D}}{l q d}\right)} \right\} \\ &+ O \left( \frac{1}{f t} \int_0^{g_1^{-1}\left(\frac{Y \sqrt{D}}{l q d}\right)} \left| \frac{d}{dw} (g_1(w) g_1'(w)) \right| + \left| \frac{d}{dw} (g_2(w) g_2'(w)) \right| dw \right). \end{aligned}$$

Since  $g_i, g_i', g_i''$  are all positive, we may drop the absolute values in the error integral. Consequently, both the evaluation terms and the error integral are bounded by

$$(28) \quad \frac{1}{f t} (g_1(w) g_1'(w) + g_2(w) g_2'(w)) \Big|_{w=g_i^{-1}\left(\frac{Y \sqrt{D}}{l q d}\right)} \ll \frac{f_1^2 Y^{\frac{3}{2}} D^{\frac{3}{4}}}{f^2 l q^2 d^2}.$$

Now we treat the integration against  $dE$ . Integrating by parts,

$$\begin{aligned}
& \int_{\max(\frac{1}{q} \frac{t^2 f_1^2}{d Y^2}, 8 f d s^2 f_1 t)}^{\frac{Y \sqrt{D}}{q l d}} z \left[ e \left( f \sqrt{\frac{z q d}{l f_1^2 t^2} - \frac{1}{Y^2}} \right) - e \left( f \sqrt{\frac{z q d}{l f_1^2 t^2} - \frac{D}{l^2 q^2 d^2 z^2}} \right) \right] d(E(z; t, s^2 f_1^2, l)) \\
(29) \quad &= O \left( \frac{Y D^{\frac{1}{2} + \epsilon}}{l q d} f_1 s t^{\frac{1}{2}} \right) - \int_{\max(\frac{1}{q} \frac{t^2 f_1^2}{d Y^2}, 8 f d s^2 f_1 t)}^{\frac{Y \sqrt{D}}{q l d}} z E(z) d \left[ e \left( f \sqrt{\frac{z q d}{l f_1^2 t^2} - \frac{1}{Y^2}} \right) - e \left( f \sqrt{\frac{z q d}{l f_1^2 t^2} - \frac{D}{l^2 q^2 d^2 z^2}} \right) \right]
\end{aligned}$$

By linearity, we can write the integral as  $I_1 - I_2$ , integrating against the two exponentials separately. In  $I_1$  put  $z = g_1(w)$  to obtain

$$I_1 = \frac{\pi i f q d}{l f_1^2 t^2} \int_{g_1^{-1}(\max(\frac{1}{q} \frac{t^2 f_1^2}{d Y^2}, 8 f d s^2 f_1 t))}^{g_1^{-1}(\frac{Y \sqrt{D}}{l q d})} g_1(w) E(g_1(w)) \frac{e(f w)}{w} g_1'(w) dw.$$

In  $I_2$  put  $z = g_2(w)$  to obtain

$$\begin{aligned}
I_2 &= \frac{\pi i f q d}{l f_1^2 t^2} \int_{g_2^{-1}(\max(\frac{D^{\frac{1}{3}} t^{\frac{2}{3}} f_1^{\frac{2}{3}}}{l^{\frac{1}{3}} q d}, 8 f d s^2 f_1 t))}^{g_2^{-1}(\frac{Y \sqrt{D}}{l q d})} g_2(w) E(g_2(w)) \frac{e(f w)}{w} g_2'(w) dw \\
&\quad + \frac{3 D \pi i f}{l^2 q^2 d^2} \int_{g_2^{-1}(\max(\frac{D^{\frac{1}{3}} t^{\frac{2}{3}} f_1^{\frac{2}{3}}}{l^{\frac{1}{3}} q d}, 8 f d s^2 f_1 t))}^{g_2^{-1}(\frac{Y \sqrt{D}}{l q d})} \frac{E(g_2(w))}{g_2(w)^2} \frac{e(f w)}{w} g_2'(w) dw = J_1 + J_2.
\end{aligned}$$

To bound  $I_1$  and  $J_1$ , we take absolute values and split each integral at  $w = \frac{1}{D}$ . For  $w < \frac{1}{D}$  we use the bound  $g_i'(w) \ll \frac{f_1^2 t^2 w}{q d}$ . Hence this part of the integrals contributes

$$(30) \quad O \left( \frac{f(g_1(\frac{1}{D}) + g_2(\frac{1}{D})) \sup |E|}{D} \right) = O \left( \frac{f f_1 s t^{\frac{1}{2}}}{D^{1-\epsilon}} \left( \frac{l f_1^2 t^2}{q d Y^2} + \frac{D^{\frac{1}{3}} f_1^{\frac{2}{3}} t^{\frac{2}{3}}}{l^{\frac{1}{3}} q d} \right) \right).$$

On the interval  $[\frac{1}{D}, g_i^{-1}(\frac{Y \sqrt{D}}{l q d})]$  we write  $\frac{dw}{w} = d(\log w)$  to get a bound

$$\sup |E| \frac{\pi f q d}{l f_1^2 t^2} \int_{\frac{1}{D}}^{g_i^{-1}(\frac{Y \sqrt{D}}{l q d})} g_i(w) g_i'(w) d \log(w) \ll \sup |E| \frac{f q d}{l f_1^2 t^2} \frac{Y \sqrt{D}}{l q d} g_i'(g_i^{-1}(\frac{Y \sqrt{D}}{l q d})) \log D.$$

Since  $g_i'(w) \ll \frac{l f_1^2 t^2}{q d} w$  and  $g_i^{-1}(\frac{Y \sqrt{D}}{l q d}) \ll \frac{Y^{\frac{1}{2}} D^{\frac{1}{4}}}{l f_1 t}$  this gives a bound of

$$(31) \quad O \left( \frac{s f Y^{\frac{3}{2}} D^{\frac{3}{4} + \epsilon}}{l^2 q d t^{\frac{1}{2}}} \right).$$

In the third integral, we also take absolute values and split at  $w = \frac{1}{D}$ , using the bound  $g_2'(w) \ll \frac{l f_1^2 t^2}{q d} w$  for  $w < \frac{1}{D}$ . Hence the portion of the integral with  $w < \frac{1}{D}$  contributes

$$(32) \quad \frac{3 D f f_1^2 t^2}{l q^3 d^3} \sup |E| \int_{g_2^{-1}(\frac{D^{\frac{1}{3}} t^{\frac{2}{3}} f_1^{\frac{2}{3}}}{l^{\frac{1}{3}} q d})}^{\frac{1}{D}} \frac{dw}{g_2(w)^2} \ll \frac{f f_1^{\frac{5}{3}} s t^{\frac{7}{6}}}{D^{\frac{2}{3} - \epsilon} q d l^{\frac{1}{3}}}$$



The integral with  $w > \frac{1}{D}$  we bound by

$$(33) \quad \frac{Df}{l^2 q^2 d^2} \sup \frac{|E|g'_2}{g_2^2} \int_{\frac{1}{D}}^{g_2^{-1}(\frac{Y\sqrt{D}}{lqd})} d(\log w) \ll \frac{D^{\frac{1}{2}+\epsilon} f f_1 s}{l q d}$$

The proposition follows by combining estimates (28), (29), (30), (31), (32) and (33), in view of the restrictions  $D^{-\frac{1}{6}} < Y < D^{\frac{1}{6}}$  and  $l f_1 t < Y^{\frac{3}{2}} D^{\frac{1}{4}}$ .  $\square$

Inserting the bounds from Proposition 5.4 into the sums over  $m$  and  $r$  in (25) gives a bound of

$$\ll \frac{1}{f} \sum_{f_1 f_2 = f} \frac{1}{f_1} \sum_{f_1 l t \leq Y^{\frac{3}{2}} D^{\frac{1}{4}}} \frac{1}{t} \sum_{s < Z} \frac{1}{s^2} \sum_{d \leq \frac{Y\sqrt{D}}{l}} \sum_{q|l} \left\{ \frac{Y^{\frac{3}{2}} D^{\frac{3}{4}+\epsilon} f_1^2}{f^2 l q^2 d^2} + \frac{Y D^{\frac{1}{2}+\epsilon} f_1 s^{\frac{1}{2}}}{l q d} + \frac{f Y^{\frac{3}{2}} D^{\frac{3}{4}+\epsilon} s}{l^2 q d t^{\frac{1}{2}}} \right\}.$$

Summed, this completes the proof of Proposition 5.3.

## 6. VERTICAL EQUIDISTRIBUTION OF POINTS ASSOCIATED TO $H_k(-d)^*$

Recall that in the introduction we defined  $H_k(-d)^*$  to be the collection of ideals of order exactly  $k$  in the class group  $H(-d)$ . In this section we partially extend the results of Section 1.2 to primitive ideals of  $H_k(-d)^*$  for odd  $k \geq 3$ .

We begin with the following generalization of the upper bound in Proposition 4.3.

**Proposition 6.1.** *For odd  $k \geq 1$  and any  $\epsilon > 0$  we have the bound*

$$\sum_{d \leq D}^* \sum_{\substack{(1) \neq \mathfrak{a} \text{ primitive} \\ [\mathfrak{a}^k] = [(1)] \\ N\mathfrak{a} \leq Y\sqrt{d}}} 1 \ll_{\epsilon} \begin{cases} Y D^{\max(1, \frac{k}{4}) + \epsilon} & Y \geq 1 \\ Y D^{1+\epsilon} + Y^{1+\frac{k}{2}} D^{\frac{k}{4}+\epsilon} & Y < O(1) \\ 0 & Y < \min(D^{-\frac{1}{2}+\frac{1}{k}}, 1) \\ Y^3 & k = 1 \end{cases}.$$

The bound for large  $Y$  is non-trivial only for  $k \leq 5$ .

Next we prove an approximate smoothed generalization of Theorem 1.2.

**Proposition 6.2.** *Let  $k > 1$  be odd and  $\phi, \psi$  non-negative functions in  $C^\infty(\mathbb{R}^+)$ ,  $\phi$  compactly supported,  $\psi$  Schwartz class with Mellin transform  $\hat{\psi}$  entire except for possibly a simple pole at zero. For  $D^{-\frac{1}{2}+\frac{1}{k}} < Y < D^{O(1)}$  we have*

$$\begin{aligned} S'_k(D, Y; \phi, \psi) &= \sum_d^* \phi\left(\frac{d}{D}\right) \sum_{\substack{(1) \neq \mathfrak{a} \text{ primitive} \\ [\mathfrak{a}^k] = [(1)] \\ N\mathfrak{a} \leq Y\sqrt{d}}} \psi\left(\frac{N\mathfrak{a}}{Y\sqrt{d}}\right) = \frac{6}{\pi^3} \hat{\phi}(1) \hat{\psi}(1) Y D + C_{1,k} \hat{\phi}\left(\frac{1}{2} + \frac{1}{k}\right) \text{Res}_{z=0} \hat{\psi}(z) D^{\frac{1}{2}+\frac{1}{k}} \\ &\quad + o(D^{\frac{1}{2}+\frac{1}{k}}) + O(Y^{1+\frac{k}{4}} D^{\frac{1}{2}+\frac{k}{8}+\epsilon}) + O((1+Y) Y^{\frac{k}{2}} D^{\frac{k}{4}+\epsilon}) + O(Y^{\frac{1}{2}} D^{\frac{3}{4}+\epsilon}). \end{aligned}$$

Here  $C_{1,k}$  is the constant from Theorem 1.4.

Finally we deduce Theorems 1.4 and 1.7 from these two propositions<sup>10</sup>.

<sup>10</sup>Note that  $S'_k(D, Y; \phi, \psi)$  defined in Proposition 6.2 includes primitive ideals  $\mathfrak{a} \neq (1)$  of order dividing  $k$ , whereas  $S_k(D, Y; \phi, \psi)$  from Theorem 1.4 does not.

Both Proposition 6.1 and 6.2 are derived from the parameterization in Proposition 3.1. Specifically,

$$(34) \quad S'_k(D, Y; \phi, \psi) = 2 \sum_{\substack{d \equiv 2 \pmod{4} \\ \text{squarefree}}} \phi\left(\frac{d}{D}\right) S(d, Y; \psi); \quad S(d, Y; \psi) = \sum_{\substack{(l, m, n, t) \in (\mathbb{Z}^+)^4 \\ lm^k = l^2 n^2 + t^2 d \\ l|d, l \text{ squarefree} \\ (m, nt) = 1}} \psi\left(\frac{lm}{Y\sqrt{d}}\right).$$

Proposition 6.1 asserts an upper bound for  $S'_k(D, Y; \chi_{[0,1]}, \chi_{[0,1]})$ , which we achieve by relaxing some of the conditions in the sum and replacing the indicator functions with smooth functions  $\phi, \psi > \chi_{[0,1]}$ . For Proposition 6.2 we write

$$(35) \quad S'_k(D, Y; \phi, \psi) = 2 \sum_{d \equiv 2 \pmod{4}} \phi\left(\frac{d}{D}\right) S(d, Y; \psi) \left\{ \sum_{s^2 | d, s < Z} \mu(s) + \sum_{s^2 | d, s > Z} \mu(s) \right\} = 2S_1 + 2S_2.$$

As in our proof of Theorem 1.2, we estimate  $S_1$  and bound  $S_2$ .

**Proposition 6.3.** For  $D^{\frac{-1}{2} + \frac{1}{k}} < Y < D^{O(1)}$  and  $1 < Z < D^{O(1)}$  the main term  $S_1$  is given by

$$S_1(D, Y; \phi, \psi) = \frac{3}{\pi^3} \hat{\phi}(1) \hat{\psi}(1) Y D + \frac{1}{2} C_{1,k} \hat{\phi}\left(\frac{1}{2} + \frac{1}{k}\right) \text{Res}_{z=0} \hat{\psi}(z) D^{\frac{1}{2} + \frac{1}{k}} \\ + O(ZY^{1+\frac{k}{2}} D^{\frac{k}{4} + \epsilon}) + O(Y^{\frac{1}{2}} D^{\frac{3}{4} + \epsilon}) + O(Z^{-1} Y D^{1+\epsilon}) + o(D^{\frac{1}{2} + \frac{1}{k}}).$$

We can bound  $S_2$  by

$$S_2 \leq \sum_{\substack{d \equiv 2 \pmod{4} \\ q^2 | d \text{ some } q > Z}} \tau(d) \phi\left(\frac{d}{D}\right) |S(d, Y; \psi)| \ll d^\epsilon S'_2; \quad S'_2 = \sum_{\substack{d \equiv 2 \pmod{4} \\ q^2 | d \text{ some } q > Z}} \phi\left(\frac{d}{D}\right) |S(d, Y; \psi)|$$

**Proposition 6.4.** Let  $D^{\frac{-1}{2} + \frac{1}{k}} < Y < D^{O(1)}$ . For any  $\epsilon > 0$ , the error sum  $S'_2$  is bounded by

$$S'_2 \ll \frac{Y D^{1+\epsilon}}{Z} + Y^{\frac{k}{2}} D^{\frac{k}{4} + \epsilon}.$$

Proposition 6.2 follows by setting  $Z = D^{\frac{1}{2} - \frac{k}{8}} Y^{\frac{-k}{4}}$  for  $Y < D^{\frac{-1}{2} + \frac{2}{k}}$  and  $Z = 1$  otherwise.

**6.1. Preliminary lemmas.** By solving

$$d = \frac{lm^k - l^2 n^2}{t^2}$$

in expression (34),  $S'_k$  is expressed as a sum over  $l, m, n, t$  by

$$(36) \quad \sum_{\substack{(l, m, n, t) \in (\mathbb{Z}^+)^4 \\ t^2 | lm^k - l^2 n^2 \\ (...)}} \phi\left(\frac{lm^k - l^2 n^2}{Dt^2}\right) \psi\left(\frac{1}{Y} \sqrt{\frac{l^2 m^2 t^2}{lm^k - l^2 n^2}}\right),$$

with (...) standing in for certain congruence and squarefree conditions. After extending  $\phi$  to a function on all of  $\mathbb{R}$  by  $\phi(x) = 0$  for  $x < 0$  we may define  $\Phi_{D,Y} : (\mathbb{R}^+)^3 \rightarrow \mathbb{R}_{\geq 0}$  by<sup>11</sup>

$$(37) \quad \Phi_{D,Y}(x, y, z) = \phi\left(\frac{x^k - y^2}{Dz^2}\right) \psi\left(\frac{1}{Y} \sqrt{\frac{x^2 z^2}{x^k - y^2}}\right).$$

Thus

$$\phi\left(\frac{lm^k - l^2 n^2}{Dt^2}\right) \psi\left(\frac{1}{Y} \sqrt{\frac{l^2 m^2 t^2}{lm^k - l^2 n^2}}\right) = \Phi(lm, l^{\frac{k+1}{2}} n, l^{\frac{k-1}{2}} t).$$

We record several useful properties of the function  $\Phi_{D,Y}$ .

<sup>11</sup>We maintain our convention that the radical in  $\psi$  is zero where it's radicand is negative.

**Lemma 6.1.** Let  $\phi \in C_c^\infty(\mathbb{R})$  have compact support in  $\mathbb{R}^+$  and let  $\psi \in C^\infty(\mathbb{R})$  be Schwartz class. Define  $\Phi_{D,Y}$  as in equation (37) and let  $\Phi = \Phi_{1,1}$ . The Mellin transform of  $\Phi_{D,Y}$  is given in terms of the Mellin transforms  $\hat{\phi}$  and  $\hat{\psi}$  by

$$\begin{aligned}\hat{\Phi}_{D,Y}(\alpha, \beta, \gamma) &= D^{\frac{\alpha}{2} + \frac{k\beta}{4} + \frac{(k-2)\gamma}{4}} Y^{\alpha + \frac{k\beta}{2} + \frac{k\gamma}{2}} \hat{\Phi}(\alpha, \beta, \gamma), \\ \hat{\Phi}(\alpha, \beta, \gamma) &= \frac{1}{4} \frac{\Gamma(\frac{\beta}{2})\Gamma(\frac{\gamma}{2} + 1)}{\Gamma(\frac{\beta+\gamma}{2} + 1)} \hat{\phi}\left(\frac{\alpha}{2} + \frac{k\beta}{4} + \frac{(k-2)\gamma}{4}\right) \hat{\psi}\left(\alpha + \frac{k\beta}{2} + \frac{k\gamma}{2}\right).\end{aligned}$$

*Proof.*

$$\begin{aligned}\hat{\Phi}(\alpha, \beta, \gamma) &= \int_0^\infty \int_0^\infty \int_0^\infty \Phi(x, y, z) x^\alpha y^\beta z^\gamma \frac{dx}{x} \frac{dy}{y} \frac{dz}{z} \\ &= \int_0^\infty \int_0^{x^{k/2}} \int_0^\infty \phi\left(\frac{x^k - y^2}{Dz^2}\right) \psi\left(\frac{xz}{Y\sqrt{x^k - y^2}}\right) x^\alpha y^\beta z^\gamma \frac{dx}{x} \frac{dy}{y} \frac{dz}{z} \\ &= \int_0^\infty \int_0^{x^{\frac{k}{2}-1}} \int_0^\infty \phi\left(\frac{x^2(x^{k-2} - y^2)}{Dz^2}\right) \psi\left(\frac{z}{Y\sqrt{x^{k-2} - y^2}}\right) x^{\alpha+\beta} y^\beta z^\gamma \frac{dx}{x} \frac{dy}{y} \frac{dz}{z} \\ &= \int_0^\infty \int_0^{x^{\frac{k}{2}-1}} \int_0^\infty \phi\left(\frac{x^2}{Dz^2}\right) \psi\left(\frac{z}{Y}\right) x^{\alpha+\beta} y^\beta z^\gamma (x^{k-2} - y^2)^{\frac{\gamma}{2}} \frac{dx}{x} \frac{dy}{y} \frac{dz}{z} \\ &= \int_0^\infty \int_0^1 \int_0^\infty \phi\left(\frac{x^2}{Dz^2}\right) \psi\left(\frac{z}{Y}\right) x^{\alpha + \frac{k\beta}{2} + \frac{(k-2)\gamma}{2}} y^\beta z^\gamma (1 - y^2)^{\frac{\gamma}{2}} \frac{dx}{x} \frac{dy}{y} \frac{dz}{z} \\ &= \frac{1}{2} \int_0^\infty \phi\left(\frac{x^2}{D}\right) x^{\alpha + \frac{k\beta}{2} + \frac{(k-2)\gamma}{2}} \frac{dx}{x} \int_0^1 y^{\frac{\beta}{2}-1} (1 - y)^{\frac{\gamma}{2}} dy \int_0^\infty \psi\left(\frac{z}{Y}\right) z^{\alpha + \frac{k\beta}{2} + \frac{k\gamma}{2}} \frac{dz}{z} \\ &= \frac{1}{4} \frac{\Gamma(\frac{\beta}{2})\Gamma(\frac{\gamma}{2} + 1)}{\Gamma(\frac{\beta+\gamma}{2} + 1)} D^{\frac{\alpha}{2} + \frac{k\beta}{4} + \frac{(k-2)\gamma}{4}} Y^{\alpha + \frac{k\beta}{2} + \frac{k\gamma}{2}} \hat{\phi}\left(\frac{\alpha}{2} + \frac{k\beta}{4} + \frac{(k-2)\gamma}{4}\right) \hat{\psi}\left(\alpha + \frac{k\beta}{2} + \frac{k\gamma}{2}\right).\end{aligned}$$

□

**Lemma 6.2.** Write  $s = \sigma + i\tau$  and  $\beta = \sigma_B + i\tau_B, \gamma = \sigma_C + i\tau_C$  with  $\sigma_*, \tau_* \in \mathbb{R}$ . For  $|\tau| > 1$  and any  $A > 0$  we have

$$|\hat{\phi}(s)|, |\hat{\psi}(s)| \ll_{A, \sigma} |\tau|^{-A}.$$

Also, for  $|\tau_B| > 1, \sigma_B, \sigma_C > 0$

$$\frac{\Gamma(\frac{\beta}{2})\Gamma(\frac{\gamma}{2} + 1)}{\Gamma(\frac{\beta+\gamma}{2} + 1)} \ll_{\sigma_B, \sigma_C} \frac{1 + |\tau_C|^{\frac{1+\sigma_C}{2}}}{|\tau_B|^{1+\frac{\sigma_C}{2}}}.$$

*Proof.* The bounds for  $\hat{\phi}$  and  $\hat{\psi}$  follow by integrating by parts the definition of the Mellin transform, while the bound for the ratio of Gamma factors can be proved by using Stirling's approximation. □

The next lemma allows us to replace a congruence condition like

$$lm^k \equiv l^2 n^2 \pmod{t^2}$$

in the sum over  $n$  in (36) with a weighted sum over all  $n \in \mathbb{Z}^+$ .

**Lemma 6.3.** Let  $\phi, \psi \in C^\infty(\mathbb{R})$  be bounded functions of bounded variation, with  $\phi$  compactly supported in  $\mathbb{R}^+$ . Define  $\Phi$  as in equation (37). For any positive integer  $A$ ,  $a \pmod{A}$ , and  $l, m, d, t \in \mathbb{Z}^+$  we have

$$\sum_{\substack{n \in \mathbb{Z}^+ \\ n \equiv a \pmod{A}}} \Phi_{D,Y}(lm, l^{\frac{k+1}{2}} dn, l^{\frac{k-1}{2}} t) = \frac{1}{A} \sum_{n \in \mathbb{Z}^+} \Phi_{D,Y}(lm, l^{\frac{k+1}{2}} dn, l^{\frac{k-1}{2}} t) + O_{\phi, \psi}(\log A).$$

*Proof.* By expanding the condition  $n \equiv a \pmod A$  in additive characters,

$$\begin{aligned} \sum_{\substack{n \in \mathbb{Z}^+ \\ n \equiv a \pmod A}} \Phi_{D,Y}(lm, l^{\frac{k+1}{2}} dn, l^{\frac{k-1}{2}} t) - \frac{1}{A} \sum_{n \in \mathbb{Z}^+} \Phi_{D,Y}(lm, l^{\frac{k+1}{2}} dn, l^{\frac{k-1}{2}} t) \\ = \frac{1}{A} \sum_{1 \leq r \leq A-1} e\left(\frac{-ra}{A}\right) \sum_{n \in \mathbb{Z}^+} \phi\left(\frac{lm^k - l^2 d^2 n^2}{Dt^2}\right) \psi\left(\frac{1}{Y} \sqrt{\frac{l^2 m^2 t^2}{lm^k - l^2 d^2 n^2}}\right) e\left(\frac{rn}{A}\right) \end{aligned}$$

By partial summation and integration by parts

$$\begin{aligned} (38) \quad \sum_{n \in \mathbb{Z}^+} \phi\left(\frac{lm^k - l^2 d^2 n^2}{Dt^2}\right) \psi\left(\frac{1}{Y} \sqrt{\frac{l^2 m^2 t^2}{lm^k - l^2 d^2 n^2}}\right) e\left(\frac{rn}{A}\right) \\ = \int_0^\infty \sum_{n \leq x} e\left(\frac{rn}{A}\right) d \left[ \phi\left(\frac{lm^k - l^2 d^2 x^2}{Dt^2}\right) \psi\left(\frac{1}{Y} \sqrt{\frac{l^2 m^2 t^2}{lm^k - l^2 d^2 x^2}}\right) \right] \end{aligned}$$

Using the bound<sup>12</sup>  $\sum_{n \leq x} e\left(\frac{rn}{A}\right) \ll \left\| \frac{r}{A} \right\|^{-1}$  and boundedness of  $\phi$  and  $\psi$  we deduce that (38) is bounded by

$$\begin{aligned} \ll_{\phi, \psi} \left\| \frac{r}{A} \right\|^{-1} \int_0^\infty \left| d \phi\left(\frac{lm^k - l^2 d^2 x^2}{Dt^2}\right) \right| + \int_0^\infty \left| d \psi\left(\frac{1}{Y} \sqrt{\frac{l^2 m^2 t^2}{lm^k - l^2 d^2 x^2}}\right) \right| \leq \left\| \frac{r}{A} \right\|^{-1} (\text{T.V.}(\phi) + \text{T.V.}(\psi)) \\ \ll_{\phi, \psi} \left\| \frac{r}{A} \right\|^{-1}. \end{aligned}$$

The proof follows because  $\frac{1}{A} \sum_{1 \leq r \leq A-1} \left\| \frac{r}{A} \right\|^{-1} \ll \log A$ .  $\square$

We also use the following second moment estimate for Dirichlet L-functions of a real character, which follows from Heath-Brown's fourth moment estimate in [9].

**Lemma 6.4.** *Let  $S(Q)$  denote the collection of real primitive Dirichlet characters of conductor at most  $Q$ , and let  $s = \sigma + it$  with  $\sigma \geq \frac{1}{2}$ . We have*

$$\sum_{\chi \in S(Q)} |L(s, \chi)|^2 \ll Q^{1+\epsilon} (1 + |t|)^{\frac{1}{2}+\epsilon}.$$

**6.2. The upper bound, proof of Proposition 6.1.** In the case  $k = 1$ , the bound

$$\sum_{d \leq D}^* \sum_{\substack{(1) \neq \mathfrak{a} \text{ primitive} \\ [\mathfrak{a}] = [(1)] \\ N\mathfrak{a} \leq Y\sqrt{d}}} 1 = O(Y^3)$$

was proven in Lemma 4.1. For  $Y > 1$ , Corollary 2.1 implies

$$\sum_{d \leq D}^* \sum_{\substack{(1) \neq \mathfrak{a} \text{ primitive} \\ [\mathfrak{a}] = [(1)] \\ N\mathfrak{a} \leq Y\sqrt{d}}} 1 \ll Y \sum_{d \leq D}^* |\{[\mathfrak{a}] \in H(-d) \text{ principal}\}| \asymp YD$$

while for  $Y \leq 1$

$$\sum_{d \leq D}^* \sum_{\substack{(1) \neq \mathfrak{a} \text{ primitive} \\ [\mathfrak{a}] = [(1)] \\ N\mathfrak{a} \leq Y\sqrt{d}}} 1 = 0$$

since any principal ideal  $\mathfrak{a}$  of  $\mathbb{Z}[\sqrt{-d}]$  has  $N\mathfrak{a} \geq d > \sqrt{d}$ .

<sup>12</sup>By  $\|\alpha\|$  we mean the distance from  $\alpha$  to the nearest integer.

When  $k > 1$  and  $Y < D^{\frac{-1}{2} + \frac{1}{k}} < 1$  we have

$$\sum_{d \leq D}^* \sum_{\substack{(1) \neq \mathfrak{a} \text{ primitive} \\ [\mathfrak{a}^k] = [(1)] \\ N\mathfrak{a} \leq Y\sqrt{d}}} 1 = 0$$

since  $[\mathfrak{a}^k] = [(1)]$  implies  $\mathfrak{a}^k = (x + y\sqrt{-d})$ ,  $y \neq 0$  so  $N(\mathfrak{a})^k = N(\mathfrak{a}^k) \geq d$  and  $N\mathfrak{a} \geq d^{\frac{1}{k}} > Y\sqrt{d}$ . The case  $Y > 1$  reduces to the case  $Y = O(1)$  by another application of Corollary 2.1.

$$\sum_{d \leq D}^* \sum_{\substack{(1) \neq \mathfrak{a} \text{ primitive} \\ [\mathfrak{a}^k] = [(1)] \\ N\mathfrak{a} \leq Y\sqrt{d}}} 1 \ll Y \sum_{d \leq D}^* \sum_{\substack{[\mathfrak{a}] \in H(-d) \\ [\mathfrak{a}]^k = [(1)]}} 1 = Y \sum_{d \leq D}^* \sum_{\substack{\mathfrak{a} \text{ primitive} \\ [\mathfrak{a}^k] = [(1)] \\ z_{\mathfrak{a}} \in \mathcal{F}}} 1 \ll YD + \sum_{d \leq D}^* \sum_{\substack{(1) \neq \mathfrak{a} \text{ primitive} \\ [\mathfrak{a}^k] = [(1)] \\ N\mathfrak{a} \leq \frac{2}{\sqrt{3}}\sqrt{d}}} 1.$$

The remaining case, to establish

$$\sum_{d \leq D}^* \sum_{\substack{(1) \neq \mathfrak{a} \text{ primitive} \\ [\mathfrak{a}^k] = [(1)] \\ N\mathfrak{a} \leq Y\sqrt{d}}} 1 \ll_{\epsilon} YD^{1+\epsilon} + Y^{\frac{k}{2}+1} D^{\frac{k}{4}+\epsilon}$$

for  $D^{\frac{-1}{2} + \frac{1}{k}} \leq Y < O(1)$  and odd  $k \geq 3$ ,<sup>13</sup> is a consequence of the following proposition.

**Proposition 6.5.** *Let  $k \geq 3$ ,  $\phi, \psi \in C^{\infty}(\mathbb{R}^+)$  non-negative,  $\phi$  with compact support and  $\psi$  Schwartz class, with Mellin transform entire except for possibly a simple pole at zero. Uniformly for fixed  $k, \phi, \psi$ ,  $D \geq 2$ ,  $0 < Y < O(1)$  and  $\epsilon > 0$  we have*

$$S'_k(D, Y; \phi, \psi) \ll YD^{1+\epsilon} + Y^{\frac{k}{2}+1} D^{\frac{k}{4}+\epsilon}.$$

To deduce the desired bound from this proposition, fix  $\phi > \chi_{[1,2]}$  and  $\psi > \chi_{[0,1]}$ . Then

$$\begin{aligned} \sum_{d \leq D}^* \sum_{\substack{(1) \neq \mathfrak{a} \text{ primitive} \\ [\mathfrak{a}^k] = [(1)] \\ N\mathfrak{a} \leq Y\sqrt{d}}} 1 &\leq \sum_{1 \leq 2^e \leq D} \sum_d^* \phi\left(\frac{d}{2^e}\right) \sum_{\substack{(1) \neq \mathfrak{a} \text{ primitive} \\ [\mathfrak{a}^k] = [(1)]}} \psi\left(\frac{N\mathfrak{a}}{Y\sqrt{d}}\right) = \sum_{1 \leq 2^e \leq D} S'_k(2^e, Y; \phi, \psi) \\ &\ll \sum_{1 \leq 2^e \leq D} \left( Y 2^{(1+\epsilon)e} + Y^{\frac{k}{2}+1} 2^{(1+\epsilon)\frac{ke}{4}} \right) \ll YD^{1+\epsilon} + Y^{\frac{k}{2}+1} D^{\frac{k}{4}+\epsilon}. \end{aligned}$$

6.2.1. *Proof of Proposition 6.5.* By (34),

$$(39) \quad S'_k(D, Y; \phi, \psi) \leq 2 \sum_{d \equiv 2 \pmod{4}} \phi\left(\frac{d}{D}\right) S(d, Y; \psi) \leq 2 \sum_d \phi\left(\frac{d}{D}\right) S'(d, Y; \psi) = 2S^+$$

with

$$S'(d, Y; \psi) = \sum_{\substack{(l, m, n, t) \in (\mathbb{Z}^+)^4 \\ lm^k = l^2 n^2 + t^2 d \\ (m, t) = (l, t) = 1}} \psi\left(\frac{lm}{Y\sqrt{d}}\right).$$

Recalling the definition of  $\Phi_{D,Y}(x, y, z)$  from expression (37) we write  $S^+$  as a sum of  $l, m, n, t$  by

$$S^+ = \sum_{\substack{(l, m, n, t) \in (\mathbb{Z}^+)^4 \\ (m, t) = (l, t) = 1 \\ lm^k \equiv l^2 n^2 \pmod{t^2}}} \Phi(lm, l^{\frac{k+1}{2}} n, l^{\frac{k-1}{2}} t).$$

<sup>13</sup>For  $k = 3$  this was already established in Proposition 4.3.

By Mellin inversion, for  $\sigma_A, \sigma_B, \sigma_C > 1$  this is given by

$$(40) \quad \sum_{\substack{(l,m,n,t) \in (\mathbb{Z}^+)^4 \\ (m,t)=(l,t)=1 \\ lm^k \equiv l^2 n^2 \pmod{t^2}}} \int_{(\sigma_C)} \int_{(\sigma_B)} \int_{(\sigma_A)} \frac{D^{\frac{\alpha}{2} + \frac{k\beta}{4} + \frac{(k-2)\gamma}{4}} Y^{\alpha + \frac{k\beta}{2} + \frac{k\gamma}{2}} \hat{\Phi}(\alpha, \beta, \gamma)}{l^{\alpha + \frac{k+1}{2}\beta + \frac{k-1}{2}\gamma} m^{\alpha} n^{\beta} t^{\gamma}} d\alpha d\beta d\gamma,$$

where, as in Lemma 6.1, we set  $\Phi = \Phi_{1,1}$ . The above expression is absolutely convergent by Lemma 6.2.

We now consider two cases for the size of  $Y$ .

Case 1:  $Y < D^{\frac{1}{2} + \frac{1}{k}}$ . In this case the sum is very small, since setting  $\sigma_A = \sigma_B = \sigma$ ,  $\sigma = 1 + (\log D)^{-1}$  and  $\sigma_C = C > 1$  gives the bound

$$(41) \quad \begin{aligned} \mathcal{S}'_k(D, Y; \phi, \psi) &\ll \int_{(C)} \int_{(\sigma)} \int_{(\sigma)} |\hat{\Phi}(\alpha, \beta, \gamma)| \sum_{l,m,n,t>0} \frac{1}{l^{k+1} m^{1+\frac{1}{\log D}} n^{1+\frac{1}{\log D}} t^{1+\frac{1}{\log D}}} d|\alpha| d|\beta| d|\gamma| \\ &\ll_C Y D^{1+\epsilon} (Y D^{\frac{1}{2} - \frac{1}{k}})^{(C+1)\frac{k}{2}}. \end{aligned}$$

Case 2:  $Y \geq D^{\frac{1}{2} + \frac{1}{k}}$ . We see that the contribution to  $\mathcal{S}^+$  from  $lm > Y D^{\frac{1}{2} + \delta}$  or  $l^{\frac{k+1}{2}} t > Y^{\frac{k}{2}} D^{\frac{(k-2)}{4} + \delta}$  is  $O_\delta(1)$ , by shifting either the  $\alpha$  or the  $\gamma$  contour rightward in expression (40). Truncating the sums accordingly and applying Lemma 6.3, we find

$$\begin{aligned} O(1) + \mathcal{S}^+ &= \sum_{\substack{(l,m,t) \in (\mathbb{Z}^+)^3 \\ (m,t)=(l,t)=1 \\ lm \leq Y D^{\frac{1}{2} + \delta} \\ l^{\frac{k+1}{2}} t \leq Y^{\frac{k}{2}} D^{\frac{k-2}{4} + \delta}}} \sum_{n^2 \equiv l^{-1} m^k \pmod{t^2}} \Phi(lm, l^{\frac{k+1}{2}} n, l^{\frac{k-1}{2}} t) \\ &= \sum_{\substack{(l,m,t) \in (\mathbb{Z}^+)^3 \\ (m,t)=(l,t)=1 \\ lm \leq Y D^{\frac{1}{2} + \delta} \\ l^{\frac{k+1}{2}} t \leq Y^{\frac{k}{2}} D^{\frac{k-2}{4} + \delta}}} \left[ \frac{\rho_{l,m}(t^2)}{t^2} \sum_{n \in \mathbb{Z}^+} \Phi(lm, l^{\frac{k+1}{2}} n, l^{\frac{k-1}{2}} t) + O(\rho_{l,m}(t^2)) \right]. \end{aligned}$$

where

$$\rho_{l,m}(t^2) = |\{n \pmod{t^2} : n^2 \equiv l^{-1} m^k \pmod{t^2}\}|.$$

For odd  $p$ ,  $\rho_{l,m}(p^\alpha) = 1 + \left(\frac{lm}{p}\right)$ , where  $\left(\frac{\cdot}{p}\right)$  is the Legendre symbol. Also  $\rho_{l,m}(2^a) \leq 4$ . Hence, if we let  $2^a \parallel t^2$ , by the Chinese Remainder Theorem,

$$\rho_{l,m}(t^2) = \rho_{l,m}(2^a) \prod_{p|t \text{ odd}} \left(1 + \left(\frac{lm}{p}\right)\right) = O(2^{\omega(t)}) = O(D^\delta)$$

so that

$$(42) \quad \mathcal{S}^+ \ll Y^{1+\frac{k}{2}} D^{\frac{k}{4}+3\delta} + D^\delta \sum_{(l,m,t) \in (\mathbb{Z}^+)^3} \frac{1}{t^2} \sum_{n \in \mathbb{Z}^+} \Phi(lm, l^{\frac{k+1}{2}} n, l^{\frac{k-1}{2}} t)$$

By Mellin inversion, for  $\sigma_A, \sigma_B, \sigma_C > 1$  the sum is equal to

$$\left(\frac{1}{2\pi i}\right)^3 \int_{(\sigma_C)} \int_{(\sigma_B)} \int_{(\sigma_A)} D^{\frac{\alpha}{2} + \frac{k\beta}{4} + \frac{(k-2)\gamma}{4}} Y^{\alpha + \frac{k\beta}{2} + \frac{k\gamma}{2}} \hat{\Phi}(\alpha, \beta, \gamma) \zeta(\alpha) \zeta(\beta) \zeta(2+\gamma) \zeta\left(\alpha + \frac{k+1}{2}\beta + \frac{k-1}{2}\gamma\right) d\alpha d\beta d\gamma$$

Shift the  $\beta$  contour to the imaginary axis, passing a pole of  $\zeta$  at  $\beta = 1$ . The function  $\hat{\Phi}(\alpha, \beta, \gamma)$  has a pole at 0 coming from the factor of  $\Gamma(\frac{\beta}{2})$ , but we can avoid this with a semicircle in the positive real direction of radius  $\frac{1}{\log D}$  about 0. To bound the resulting integral, set  $\sigma_A = \sigma_C = 1 + \frac{1}{\log D}$  and note that the resulting integral narrowly remains absolutely convergent, because  $|\zeta(\beta)| \ll \log D + |\beta|^{\frac{1}{2}} \log |\beta|$  whereas the ratio of Gamma functions in  $\hat{\Phi}(\alpha, \beta, \gamma)$  decays like  $(1 + |\beta|)^{\frac{-3}{2} - \frac{\log D}{2}}$ . The error integral is thus bounded by

$$(43) \quad O(Y^{1+\frac{k}{2}} D^{\frac{k}{4}+\delta}).$$

In the polar term, there are no longer issues of convergence. We set the  $\alpha$  contour at  $\Re(\alpha) = 1 + \frac{1}{\log D}$  and the  $\gamma$  contour at  $\Re(\gamma) = -1 + \frac{1}{\log D}$  to get a bound of size

$$(44) \quad O(YD^{1+\delta}).$$

Inserting estimates (43) and (44) in expression (42) completes the proof of Proposition 6.5.

**6.3. The main term  $\mathcal{S}_1$ , proof of Proposition 6.3.** The proof is quite similar to that of Proposition 6.5, but we take a bit of extra care to extract the main term.

By solving  $d = \frac{lm^k - l^2 n^2}{t^2}$  as in equation (34),

$$\mathcal{S}_1 = \sum_{s < Z \text{ odd}} \mu(s) \sum_{\substack{d \equiv 2 \pmod{4} \\ s^2 | d}} \phi\left(\frac{d}{s}\right) S(d, Y; \psi) = \sum_{s < Z \text{ odd}} \mu(s) \sum_{\substack{(l, m, n, t) \in (\mathbb{Z}^+)^4 \\ (m, 2lnt) = (l, st) = 1 \\ l \text{ squarefree} \\ lm^k - l^2 n^2 \equiv 2s^2 t^2 \pmod{4s^2 t^2}}} \Phi_{D, Y}(lm, l^{\frac{k+1}{2}} n, l^{\frac{k-1}{2}} t).$$

Truncating the sum at  $lm \leq YD^{\frac{1}{2}+\delta}$ ,  $l^{\frac{k-1}{2}} t < Y^{\frac{k}{2}} D^{\frac{k-2}{4}+\delta}$ , setting apart the sum over  $n$ , and applying Möbius inversion to eliminate the condition  $(m, n) = 1$  we obtain

$$(45) \quad \mathcal{S}_1 + O(1) = \sum_{s < Z \text{ odd}} \mu(s) \sum_{\substack{(l, m, t) \in (\mathbb{Z}^+)^3 \\ (m, 2lst) = (l, st) = 1 \\ l \text{ squarefree} \\ lm < YD^{\frac{1}{2}+\delta} \\ l^{\frac{k-1}{2}} t < Y^{\frac{k}{2}} D^{\frac{k-2}{4}+\delta}}} \sum_{d|m} \mu(d) \sum_{\substack{n: lm^k - l^2 d^2 n^2 \\ \equiv 2s^2 t^2 \pmod{4s^2 t^2}}} \Phi_{D, Y}(lm, l^{\frac{k+1}{2}} dn, l^{\frac{k-1}{2}} t).$$

For odd  $m$  and any  $d|m$ , set  $\rho_{l, m}(r) = |\{n \pmod{4r} : lm^k - l^2 d^2 n^2 \equiv 2r \pmod{4r}\}|$ .

**Lemma 6.5.** *Let  $(lm, r) = 1$  with  $m$  odd,  $d|m$  and say  $2^a || r$ . We have*

$$\rho_{l, m}(r) = \rho_{l, m}(2^a) \prod_{p|r \text{ odd}} \left(1 + \left(\frac{lm}{p}\right)\right); \quad \rho_{l, m}(2^a) = \begin{cases} 4 & l \text{ even} \\ 2 & a = 0, lm \equiv 3 \pmod{4} \\ 4 & a \geq 1, lm \equiv 2^{a+1} + 1 \pmod{8} \end{cases}$$

*Proof.* See Lemma 4.2. □

In particular,  $\rho_{l, m}(s^2 t^2) = O(2^{\omega(st)}) = O(D^\delta)$ . Hence, applying Lemma (6.3) to the sum over  $n$  in (45), bounding the number of divisors of  $m$  by  $O(D^\delta)$ , then removing the bounds on the sums over  $l, m, t$ , we obtain

$$(46) \quad \mathcal{S}_1 = O(ZY^{\frac{k}{2}+1} D^{\frac{k}{4}+4\delta}) + \frac{1}{4} \sum_{s < Z \text{ odd}} \frac{\mu(s)}{s^2} \sum_{\substack{(l, m, t) \in (\mathbb{Z}^+)^3 \\ (m, 2lst) = (l, st) = 1 \\ l \text{ squarefree}}} \frac{\rho_{l, m}(s^2 t^2)}{t^2} \sum_{d|m} \mu(d) \sum_n \Phi_{D, Y}(lm, l^{\frac{k+1}{2}} dn, l^{\frac{k-1}{2}} t).$$

Dropping the error and applying Mellin inversion, then shifting the sum over  $n$  under the integral, we find that for  $\sigma_A, \sigma_B, \sigma_C > 1$

$$\begin{aligned} \mathcal{S}_1 &= \frac{1}{4} \sum_{s < Z \text{ odd}} \frac{\mu(s)}{s^2} \sum_{\substack{(l, m, t) \in (\mathbb{Z}^+)^4 \\ (m, 2lst) = (l, st) = 1 \\ l \text{ squarefree} \\ 2^a || t^2}} \rho_{l, m}(2^a) \sum_{\substack{q|st \text{ odd} \\ \text{squarefree}}} \left(\frac{lm}{q}\right) \sum_{d|m} \mu(d) \\ &\quad \left(\frac{1}{2\pi i}\right)^3 \int_{(\sigma_A)} \int_{(\sigma_B)} \int_{(\sigma_C)} \frac{D^{\frac{\alpha}{2} + \frac{k\beta}{4} + \frac{(k-2)\gamma}{4}} \gamma^{\alpha + \frac{k\beta}{2} + \frac{k\gamma}{2}} \hat{\Phi}(\alpha, \beta, \gamma) \zeta(\beta)}{l^{\alpha + \frac{k+1}{2}\beta + \frac{k-1}{2}\gamma} m^\alpha d^\beta t^{2+\gamma}} d\alpha d\beta d\gamma. \end{aligned}$$

Shifting the  $\beta$  contour to  $\Re(\beta) = 0$  we pass a pole of  $\zeta$  at  $\beta = 1$ . The error integral can be bounded by shifting the  $\alpha$  and  $\gamma$  contours to  $1 + \frac{1}{\log D}$ , giving a bound of

$$(47) \quad O(Y^{1+\frac{k}{2}} D^{\frac{k}{4}+\delta}).$$

Write  $\chi_q$  for the quadratic character  $\left(\frac{\cdot}{q}\right)$ . Also, set

$$c(l, t) = \begin{cases} 4 & l \text{ even, } t \text{ odd} \\ 2 & lt \text{ odd} \\ 4 & l \text{ odd, } t \text{ even} \end{cases}, \quad r(l, t) = \begin{cases} \{1, 3, 5, 7 \bmod 8\} & l \text{ even, } t \text{ odd} \\ \{3, 7 \bmod 8\} & lt \text{ odd} \\ \{1 \bmod 8\} & l \text{ odd, } t \text{ even} \end{cases}$$

After shifting sums, the residue term is equal to

$$\frac{Y^{\frac{k}{2}} D^{\frac{k}{4}}}{4} \sum_{q \text{ odd}}^b \sum_{\substack{(l, s, t) \in (\mathbb{Z}^+)^3 \\ (l, st)=1 \\ l, s \text{ squarefree} \\ q|st \\ s < Z \text{ odd}}} \frac{\mu(s)c(l, t)\chi_q(l)}{s^2} \sum_{a \in r(l, t)} \frac{1}{4} \sum_{\theta \bmod 8} \theta(l)\bar{\theta}(a) \sum_{(m, 2lst)=1} \chi_q \cdot \theta(m) \sum_{d|m} \frac{\mu(d)}{d} \left(\frac{1}{2\pi i}\right)^2 \int_{(\sigma_C)} \int_{(\sigma_A)} \frac{D^{\frac{\alpha}{2} + \frac{(k-2)\gamma}{4}} Y^{\alpha + \frac{k}{2}\gamma} \hat{\Phi}(\alpha, 1, \gamma)}{l^{\alpha + \frac{k+1}{2} + \frac{k-1}{2}\gamma} m^{\alpha} t^{\gamma+2}} d\alpha d\gamma,$$

which, after passing the sums over  $m$  and  $d$  under the integral, is equal to

$$\frac{Y^{\frac{k}{2}} D^{\frac{k}{4}}}{16} \sum_{q \text{ odd}}^b \sum_{\theta \bmod 8} \sum_{\substack{(l, s, t) \in (\mathbb{Z}^+)^3 \\ (l, st)=1 \\ l, s \text{ squarefree} \\ q|st \\ s < Z \text{ odd}}} \frac{\mu(s)c(l, t)\chi_q \cdot \theta(l)}{s^2} \sum_{a \in r(l, t)} \bar{\theta}(a) \left(\frac{1}{2\pi i}\right)^2 \int_{(\sigma_C)} \int_{(\sigma_A)} \frac{D^{\frac{\alpha}{2} + \frac{(k-2)\gamma}{4}} Y^{\alpha + \frac{k}{2}\gamma} \hat{\Phi}(\alpha, 1, \gamma) L(\alpha, \chi_q \cdot \theta) \prod_{p|2lst} (1 - \chi_q \cdot \theta(p)p^{-\alpha})}{l^{\alpha + \frac{k+1}{2} + \frac{k-1}{2}\gamma} t^{\gamma+2} L(\alpha+1, \chi_q \cdot \theta) \prod_{p|2lst} (1 - \chi_q \cdot \theta(p)p^{-\alpha-1})} d\alpha d\gamma.$$

We shift the  $\alpha$  contour to  $\sigma_A = \frac{1}{2}$  passing a pole at  $\alpha = 1$  of  $L(\chi_q \cdot \theta_0, \alpha)$  when  $q = 1$ . To bound the error integral, shift the  $\gamma$  contour to  $\sigma_C = -1 + \frac{1}{\log D}$ . There are no convergence issues because  $\hat{\Phi}(\alpha, 1, \gamma)$  has rapid decay in vertical strips for both  $\alpha$  and  $\gamma$ . We have  $L(\alpha+1, \chi_q \cdot \theta)$  is bounded, while the products are bounded by  $\tau(lst)^2$ . Taking all of the sums under the integral and bounding absolutely we obtain an error which is

$$\ll Y^{\frac{1}{2}} D^{\frac{3}{4}+\delta} \int_{(\frac{1}{2})} \int_{(\sigma_C)} |\hat{\Phi}(\alpha, 1, \gamma)| \sum_q \sum_{\substack{\chi \bmod q \\ \text{real}}} \frac{|L(\alpha, \chi)|}{q^{1+\frac{1}{\log D}}} d|\alpha| d|\gamma|.$$

Bounding  $|L(\alpha, \chi)| \leq 1 + |L(\alpha, \chi)|^2$ , applying Lemma 6.4 and partial summation, the sum over  $q$  and  $\chi$  is bounded by  $(1 + |\alpha|)^{\frac{1}{2}} \log D$ . Since  $\int_{(\frac{1}{2})} \int_{(\sigma_C)} |\hat{\Phi}(\alpha, 1, \gamma)| (1 + |\alpha|)^{\frac{1}{2}} d|\alpha| d|\gamma|$  converges, the error integral is

$$(48) \quad O(Y^{\frac{1}{2}} D^{\frac{3}{4}+\delta}).$$

Set

$$c(l) = c(l, t)|r(l, t)| = \begin{cases} 16 & l \text{ even} \\ 4 & l \text{ odd} \end{cases}.$$

The residue from the  $q = 1, \theta = \theta_0$  term is equal to

$$\frac{Y^{\frac{k}{2}+1} D^{\frac{k}{4}+\frac{1}{2}}}{24\zeta(2)} \sum_{\substack{(l, s, t) \in (\mathbb{Z}^+)^3 \\ (l, st)=1 \\ l, s \text{ squarefree} \\ s < Z \text{ odd}}} \frac{\mu(s)c(l)}{s^2} \prod_{p|lst \text{ odd}} \frac{p}{p+1} \frac{1}{2\pi i} \int_{(\sigma_C)} \frac{D^{\frac{(k-2)\gamma}{4}} Y^{\frac{k\gamma}{2}} \hat{\Phi}(1, 1, \gamma)}{l^{\frac{k+3}{2} + \frac{k-1}{2}\gamma} t^{2+\gamma}} d\gamma.$$



Shift the line of integration to  $\Re(\gamma) = -1 + \frac{1}{\log D}$  and extend the sum over  $s$  to all odd squarefree numbers. The error from doing this is

$$(49) \quad O(YD^{1+\delta}Z^{-1}).$$

Now pass all of the summations under the integral. We are left to evaluate

$$\frac{Y^{\frac{k}{2}+1}D^{\frac{k}{4}+\frac{1}{2}}}{24\zeta(2)} \frac{1}{2\pi i} \int_{(\sigma_C)} Y^{\frac{k}{2}\gamma} D^{\frac{(k-2)\gamma}{4}} \hat{\Phi}(1, 1, \gamma) \zeta(2+\gamma) H(\gamma) d\gamma$$

where we have factored out  $\zeta(2+\gamma)$  from the sum to leave

$$H(\gamma) = 4 \left[ 1 + \frac{1}{2^{\frac{k-1}{2}(\gamma+1)}} - \frac{1}{2^{\frac{k+3}{2}+\frac{k+1}{2}\gamma}} \right] \prod_{p \text{ odd}} \left[ 1 - \frac{1}{p+1} \left( \frac{1}{p} + \frac{1}{p^{2+\gamma}} - \frac{1}{p^{\frac{k+1}{2}+\frac{k-1}{2}\gamma}} + \frac{1}{p^{\frac{k+5}{2}+\frac{k+1}{2}\gamma}} \right) \right].$$

For  $k \geq 3$  and  $\sigma_C > \sigma > \frac{-(k+1)}{k-1}$  the Euler product defining  $H$  is absolutely convergent and hence, uniformly bounded. At this point we recall

$$\hat{\Phi}(1, 1, \gamma) = \frac{1}{4} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{\gamma}{2}+1)}{\Gamma(\frac{\gamma}{2}+\frac{3}{2})} \hat{\Phi}(\frac{1}{2} + \frac{k}{4} + \frac{k-2}{4}\gamma) \hat{\Psi}(1 + \frac{k}{2} + \frac{k}{2}\gamma)$$

has a pole at  $-1 - \frac{2}{k}$  of residue

$$\frac{2}{k} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-\frac{1}{k})}{\Gamma(1-\frac{1}{k})} \hat{\Phi}(\frac{1}{2} + \frac{1}{k}) \text{Res}_{s=0} \hat{\Psi}(s).$$

Pushing the line of integration to  $\sigma_C = \frac{-(k+1)}{k-1} + \delta$  we pass a pole of  $\zeta$  at  $\gamma = -1$  with residue  $\frac{3}{\pi^3} YD \hat{\Phi}(1) \hat{\Psi}(1)$  and the pole of  $\hat{\Phi}(1, 1, \gamma)$  at  $\frac{-(k+2)}{k}$  with residue

$$\frac{2\sqrt{\pi}\zeta(1-\frac{2}{k})\Gamma(\frac{1}{2}-\frac{1}{k})}{24k\zeta(2)\Gamma(1-\frac{1}{k})} D^{\frac{1}{2}+\frac{1}{k}} \hat{\Phi}(\frac{1}{2} + \frac{1}{k}) \text{Res}_{s=0} \hat{\Psi}(s).$$

The remaining integral is bounded by  $o(D^{\frac{1}{2}+\frac{1}{k}})$ . This, together with estimates (46), (47), (48) and (49) completes the proof.

#### 6.4. The error term $S'_2$ , proof of Proposition 6.4.

Recall the definition

$$S'_2 = \sum_{\substack{d \equiv 2 \pmod{4} \\ q^2 | d \text{ some } q > Z}} S(d, Y; \psi); \quad S(d, Y; \psi) = \sum_{\substack{(l, m, n, t) \in (\mathbb{Z}^+)^4 \\ lm^k = l^2 n^2 + t^2 d \\ l | d, l \text{ squarefree} \\ (m, n, t, d) = 1}} \psi\left(\frac{lm}{Y\sqrt{d}}\right)$$

with  $\psi$  a Schwartz class function. To begin, we remark that it suffices to prove the bound

$$(50) \quad \sum_{\substack{d \equiv 2 \pmod{4} \\ q^2 | d \text{ some } q > Z}} S(d, Y) \ll \frac{YD^{1+\epsilon}}{Z} + Y^{\frac{k}{2}} D^{\frac{k}{4}+\epsilon}; \quad S(d, Y) = S(d, Y; \chi_{[0,1]})$$

because the growth of  $S(d, Y)$  is at most polynomial in  $Y$  while  $\psi$  decays faster than any polynomial, so that independent of  $d$  and  $Y$ ,  $S(d, Y; \psi) \ll_\epsilon S(d, YD^\epsilon)$ .

Let  $(l, m, n, t)$  be a tuple counted in (50). Since  $(m, 2lnt) = 1$  any square factor of  $d$  must be prime to  $l$ , so we may write  $d = q^2 \alpha l$  where  $\alpha l$  is squarefree,  $\alpha l \equiv 2 \pmod{4}$ . In the field  $\mathbb{Q}(\sqrt{-\alpha l})$  we have the factorization of ideals

$$(l)(m)^k = (ln + tq\sqrt{-\alpha l})(ln - tq\sqrt{-\alpha l}).$$

Note that  $(ln, tq) = 1$  so  $(ln + tq\sqrt{-\alpha l})$  is a non-trivial primitive ideal of  $\mathbb{Q}(\sqrt{-\alpha l})$ . Since  $l | \alpha l$ ,  $(l) = l^2$  with  $l$  a primitive ideal dividing the different  $\mathfrak{d}$ . Each prime in  $l$  divides  $(ln + tq\sqrt{-\alpha l})$  and  $(ln - tq\sqrt{-\alpha l})$  an equal number of times, that is, exactly once, so we may set  $\mathfrak{b} = l^{-1}(ln + tq\sqrt{-\alpha l})$ ,  $\bar{\mathfrak{b}} = l^{-1}(ln - tq\sqrt{-\alpha l})$ ,

$$(m)^k = \mathfrak{b}\bar{\mathfrak{b}}.$$

Now  $(m, \alpha l) = 1$  so  $(b, d) = (1)$ , so that  $(b, \bar{b}) = (1)$ . Moreover,  $b \neq (1)$  since otherwise  $m = 1$  in  $lm^k = l^2 n^2 + t^2 d$ , which is impossible. It follows that there exists primitive  $c$ ,  $(c, d) = (1)$  with  $b = c^k$ . Set  $a = lc$ . Then  $a$  is primitive,  $a \neq (1)$  and  $a$  satisfies

$$a^k = l^{k-1} lb = (l)^{\frac{k-1}{2}} (ln + tq\sqrt{-\alpha l})$$

is principal, that is,  $a \neq (1)$  is a primitive ideal of  $H_k(-\alpha l)$ . Note that  $Na = lm$ .

Now fix  $\alpha, l$  and suppose that  $(1) \neq a$  is a non-trivial primitive ideal obtained as above. From the norm we know  $m$ . By taking the generator of the principal ideal  $l^{-\frac{k-1}{2}} a^k = (ln + tq\sqrt{-\alpha l})$  in  $\mathbb{Q}(\sqrt{-\alpha l})$  we can deduce  $n$  and the number  $tq$ , that is, we fix the tuple  $(l, m, n, t)$  up to a divisor function. In other words, the number of tuples  $(l, m, n, t)$  mapping to  $a$  under the construction above is at most  $O(D^\epsilon)$ . We conclude

$$\begin{aligned} S'_2 &\ll D^\epsilon \sum_{\substack{\alpha l \leq \frac{D}{Z^2} \\ \alpha l \equiv 2 \pmod{4}}}^b \left| \left\{ (1) \neq a \text{ primitive}, [a^k] = [(1)], Na \ll Y\sqrt{D} \right\} \right| \\ &\ll \sum_{1 \leq 2^e \leq \frac{D}{Z^2}} \sum_{\substack{d \leq 2^{e+1} \\ d \equiv 2 \pmod{4} \\ \text{squarefree}}} \left| \left\{ (1) \neq a \text{ primitive}, [a^k] = [(1)], Na \leq \frac{Y\sqrt{D}}{2^{\frac{e}{2}}} \sqrt{d} \right\} \right|. \end{aligned}$$

By Proposition 6.1, this is bounded by

$$\begin{aligned} &\ll \sum_{1 \leq 2^e \leq \min(Y^2 D, \frac{D}{Z^2})} \frac{Y\sqrt{D}}{2^{\frac{e}{2}}} (2^e)^{\max(1, \frac{k}{4}) + \epsilon} + \sum_{Y^2 D \leq 2^e \leq \frac{D}{Z^2}} \left( \frac{Y\sqrt{D}}{2^{\frac{e}{2}}} (2^e)^{1+\epsilon} + \left( \frac{Y\sqrt{D}}{2^{\frac{e}{2}}} \right)^{1+\frac{k}{2}} (2^e)^{\frac{k}{4}+\epsilon} \right) \\ &= O\left( \frac{YD^{1+\epsilon}}{Z} + Y^{\frac{k}{2}} D^{\frac{k}{4}+\epsilon} \right). \end{aligned}$$

## 6.5. Deduction of Theorems 1.4 and 1.7.

*Proof of Theorem 1.4.* By Möbius inversion,

$$\begin{aligned} S_k(D, Y; \phi, \psi) &= \sum_{\substack{d \equiv 2 \pmod{4} \\ \text{squarefree}}} \phi\left(\frac{d}{D}\right) \sum_{a \in H_k(-d)^* \text{ primitive}} \psi\left(\frac{Na}{Y\sqrt{d}}\right) \\ &= \sum_{k' | k} \mu\left(\frac{k}{k'}\right) \sum_{\substack{d \equiv 2 \pmod{4} \\ \text{squarefree}}} \phi\left(\frac{d}{D}\right) \sum_{\substack{(1) \neq a \text{ primitive} \\ [a^{k'}] = [(1)]}} \psi\left(\frac{Na}{Y\sqrt{d}}\right) = \sum_{k' | k, k' > 1} \mu\left(\frac{k}{k'}\right) S'_k(D, Y; \phi, \psi) + O(Y^2 D^{\frac{1}{2}}), \end{aligned}$$

with the error bounding the term from  $k' = 1$ . We apply Proposition 6.2 for  $k = k'$  to get the main term, plus an error that is

$$O(Y^{\frac{k}{2}} D^{\frac{k}{4}+\epsilon}) + O(Y^{1+\frac{k}{4}} D^{\frac{1}{2}+\frac{k}{8}+\epsilon}) + O(Y^{\frac{1}{2}} D^{\frac{3}{4}+\epsilon}).$$

If this error term dominates the main term of size  $YD$  then by Proposition 6.5 it also dominates all of the terms  $1 < k' < k$ , so we may assume  $YD > Y^{\frac{k}{2}} D^{\frac{k}{4}}$ , that is,

$$D^{\frac{-1}{2} + \frac{1}{k}} \leq Y \leq D^{\frac{-1}{2} + \frac{1}{k-2}}.$$

Since  $k$  is odd, if  $k' | k$  and  $k' \neq k$  we have  $k' \leq \frac{k}{3}$ . Hence  $Y \leq D^{\frac{-1}{2} + \frac{1}{k'} - \delta}$ . It then follows by partial summation against Proposition 6.1 that  $S'_{k'}(D, Y; \phi, \psi) = o(1)$  for all  $k' | k$ ,  $k' < k$ , since  $\psi$  is Schwartz class.  $\square$

*Proof of Theorem 1.7.* The upper bound follows by taking  $Y = \frac{2}{\sqrt{3}}$  in Proposition 6.1. For the lower bound, choose  $\phi, \psi \leq \chi_{[0,1]}$  and  $Y < D^{\frac{-1}{2} + \frac{1}{k-2} - \delta}$ . Then

$$S_k(D, Y; \phi, \psi) \leq \sum_{\substack{d \leq D \\ d \equiv 2 \pmod{4}}} |H_k(-d)^*|.$$

The theorem now follows from the asymptotic in Theorem 1.4.  $\square$

## 7. A NEW PROOF OF THE DAVENPORT-HEILBRONN THEOREM

We conclude by proving Theorem 1.5, which we obtain from the following lemma.

**Lemma 7.1.** *Let  $\Psi_0(y) = (2\pi y - 1)e^{-\pi y}$  and  $\Psi(y) = \sum_{m=1}^{\infty} \Psi_0(m^2 y)$ . For any  $z \in \mathbb{H}$  we have*

$$\sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} \Psi(\mathcal{I}(\gamma \cdot z)^{-1}) = \frac{1}{2}.$$

*Proof.* Let  $E(z, s) = \sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} \mathcal{I}(\gamma \cdot z)^s$  be the non-holomorphic Eisenstein series, which satisfies the functional equation

$$E^*(z, s) = \pi^{-s} \Gamma(s) \zeta(2s) E(z, s) = E^*(z, 1 - s).$$

The Mellin transform of  $\Psi$  is

$$\hat{\Psi}(s) = (2s - 1) \pi^{-s} \Gamma(s) \zeta(2s),$$

which has rapid decay as  $|\mathcal{I}(s)| \rightarrow \infty$ . Hence, by Mellin inversion we have

$$\sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} \Psi(\mathcal{I}(\gamma \cdot z)^{-1}) = \frac{1}{2\pi i} \int_{(2)} E(z, s) \hat{\Psi}(s) ds = \frac{1}{2\pi i} \int_{(2)} (2s - 1) E^*(z, s) ds$$

Shifting the contour to  $\Re(s) = \frac{1}{2}$  we pass a pole of  $E^*(z, s)$  at  $s = 1$ , with residue  $\frac{1}{2}$ . Thus

$$\sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} \Psi(\mathcal{I}(\gamma \cdot z)^{-1}) = \frac{1}{2} + \frac{1}{2\pi i} \int_{(\frac{1}{2})} (2s - 1) E^*(z, s) ds.$$

But the integral on the half line vanishes, because substituting  $t = 1 - s$  and using the functional equation,

$$\frac{1}{2\pi i} \int_{(\frac{1}{2})} (2s - 1) E^*(z, s) ds = -\frac{1}{2\pi i} \int_{(\frac{1}{2})} (2t - 1) E^*(z, t) dt.$$

$\square$

*Proof of Theorem 1.5.* We have

$$\sum_d^* \phi\left(\frac{d}{D}\right) |H_k(-d)^*| = \sum_d^* \phi\left(\frac{d}{D}\right) \sum_{[a] \in H_k(-d)^*} \mathbf{1}(z_{[a]}).$$

By Lemma 7.1 we can write  $\mathbf{1}(z) = 2 \sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} \Psi(\mathcal{I}(\gamma \cdot z)^{-1})$ . Inserting this and exchanging order of summation, we find

$$\sum_d^* \phi\left(\frac{d}{D}\right) |H_k(-d)^*| = 2 \sum_d^* \phi\left(\frac{d}{D}\right) \sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} \sum_{[a] \in H_k(-d)^*} \Psi(\mathcal{I}(\gamma \cdot z_{[a]})^{-1}).$$

But this last sum is equal to

$$\sum_d^* \phi\left(\frac{d}{D}\right) \sum_{\substack{a \text{ primitive} \\ [a] \in H_k(-d)^*}} \Psi(\mathcal{I}(z_a)^{-1}) = S_k(D, 1; \phi, \Psi)$$

by Proposition 2.3.  $\square$

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*E-mail address:* `rdhough@stanford.edu`